Workshop on Qubits and Spacetime

Institute of Advanced Studies

December 3 - 4th, 2019

Caltech

Entanglement Wedge Reconstruction in Infinite-dimensional Hilbert Spaces

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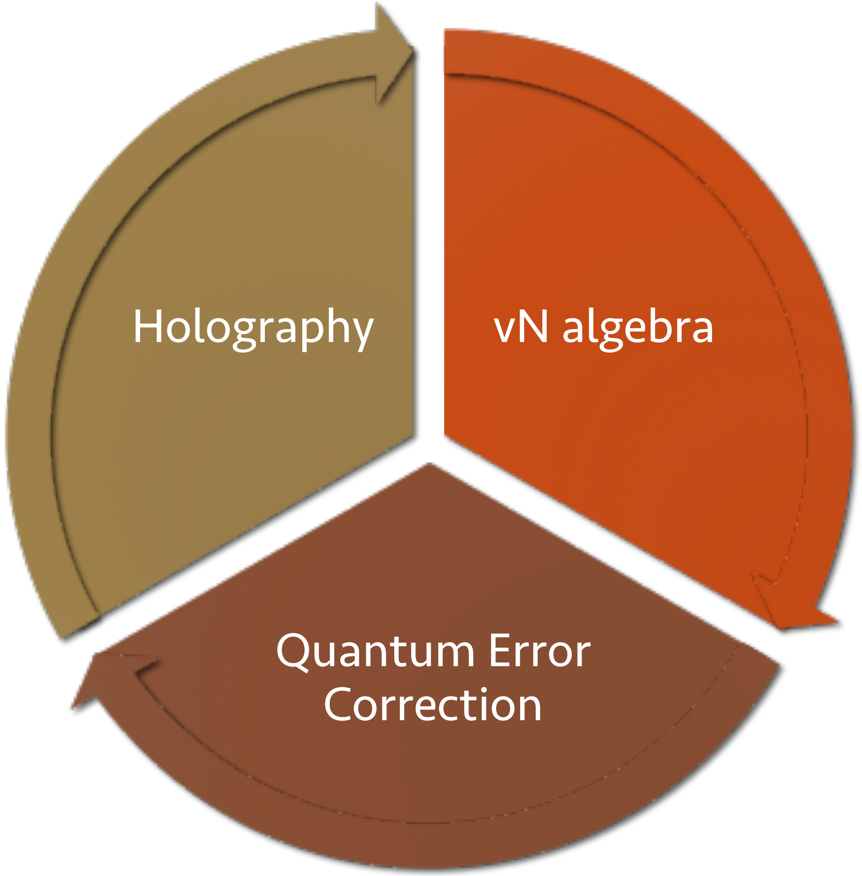
arXiv: 1811.05482, 1910.06328 (MJK, Kolchmeyer), To appear (MJK, Tang), To appear(MJK, Gesteau)

Three ingredients



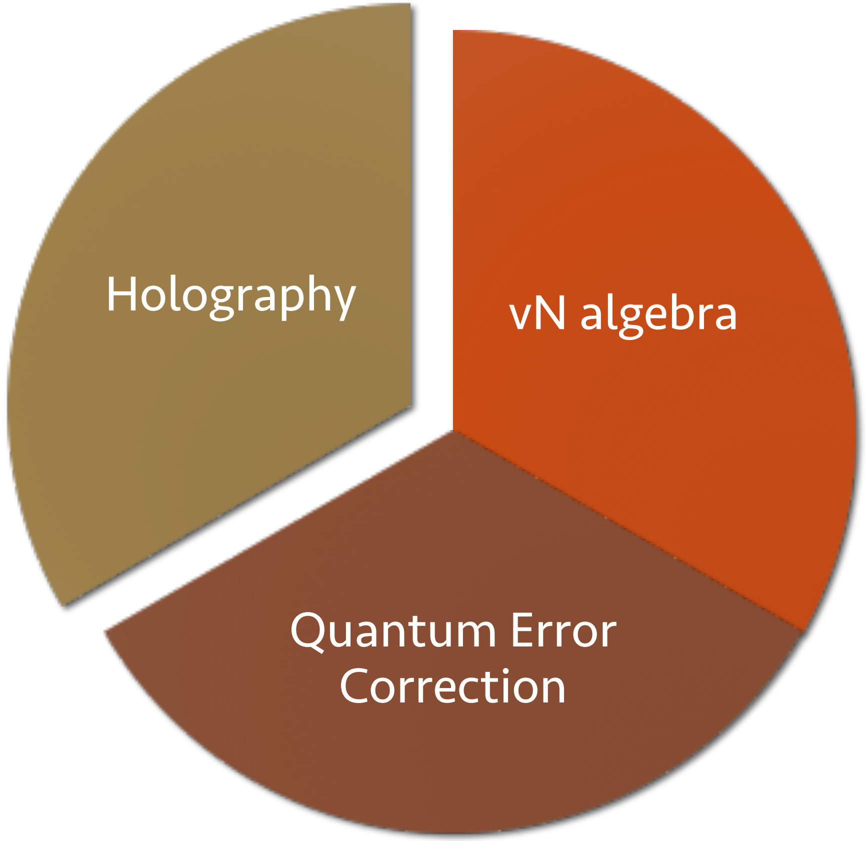
Holography

Bulk ⟷ Boundary

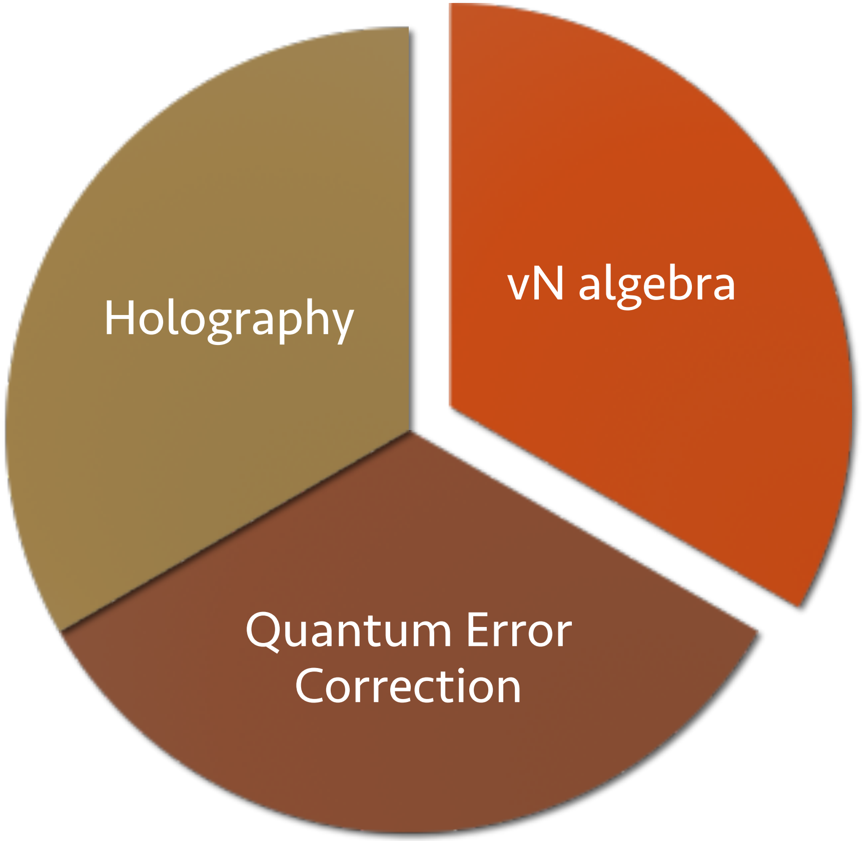
Local bulk operators

⇓

Boundary operators
smeared over the entire spatial slice
or a compact spatial subregion



Von Neumann algebra



Local operator algebra

= Von Neumann algebra

- Finite-dimensional Hilbert space: I_n
- Infinite-dimensional Hilbert space:

$$\mathrm{I}_\infty,\ \mathrm{II}_1,\ \mathrm{II}_\infty,\ \mathrm{III}_\lambda\ (0 \leq \lambda \leq 1)$$

Local operator algebra = vN algebra

- QFTs with (infinite-dimensional) Hilbert space
- Local operator algebras

  = (infinite-dimensional) von Neumann algebra $M \subset B(\mathscr{H})$

  the set of bounded operators in $\mathscr{H}$

- Commutant: $M' = \{\mathcal{O} \in B(\mathscr{H}) \,|\, \mathcal{O}\mathscr{P} = \mathscr{P}\mathcal{O} \ {}^{\forall}\mathscr{P} \in M\}$
- Von Neumann algebra:

  an algebra of bounded operators that

  1) contains the identity, 2) closed under Hermitian conjugation, 3) $M = M''$

  (**-algebra*)

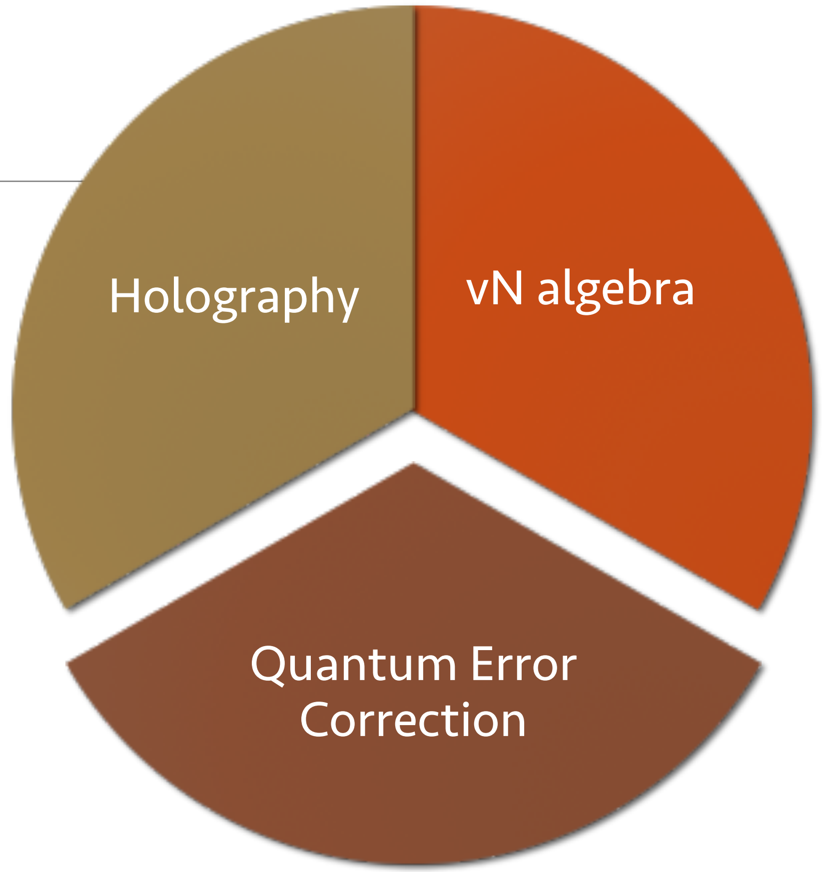
Quantum Error Correction

Local bulk operators

⇓

Code Subspace

of the physical Hilbert space
of the CFT

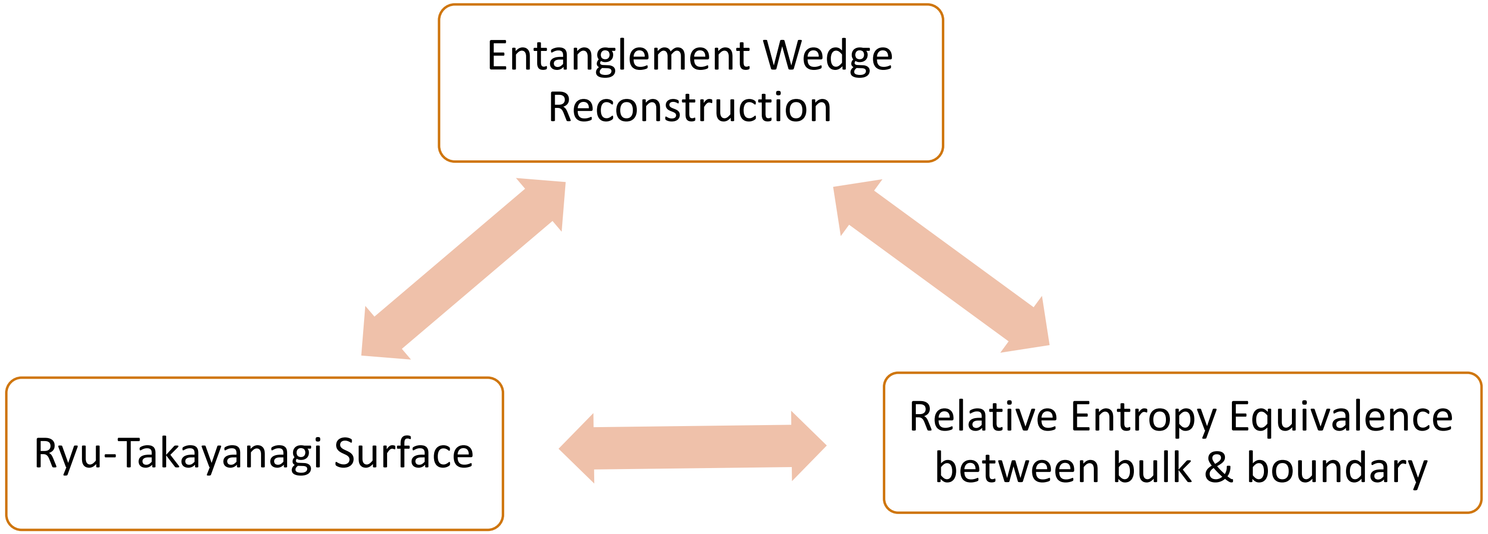


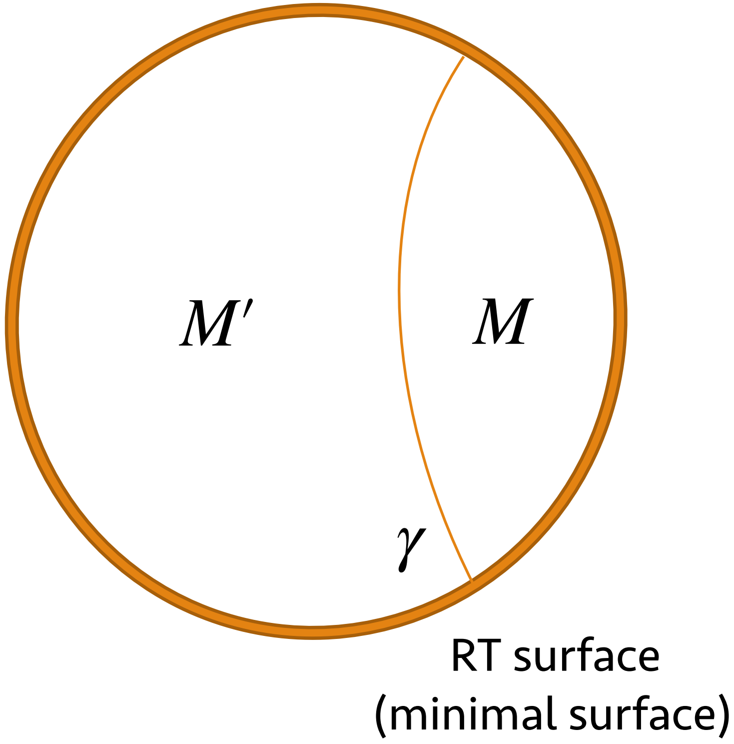
Finite-dimensional Hilbert space:

Entanglement Wedge Reconstruction ⟺ Ryu-Takayanagi Surface ⟺ Relative Entropy Equivalence Between the bulk and the boundary

Finite-dimensional Hilbert space

Using Code subspace:





Reeh-Schlieder theorem

- ➢ Want theories compatible with Reeh-Schlieder theorem.
  - Then this leads to infinite-dimensional Hilbert spaces!

- ➢ Reeh-Schlieder theorem:

  For any region A, by acting on the vacuum $|\Omega\rangle$ with operators located in that region we can produce a set of states which is dense in the full Hilbert space of the QFT.

- ➢ Start with a cyclic and separating state, by acting with a suitable local operator

  $\Longrightarrow$ Obtain a dense subset of the Hilbert space $\mathscr{H}$

Cyclic and Separating state

- Define a map $e_\Psi : M \longrightarrow \mathscr{H}$ where $\mathscr{O} \mapsto \mathscr{O}|\Psi>$
- $|\Psi>$ is cyclic with respect to a von Neumann algebra M
  $\Longleftrightarrow$ $\mathscr{H}$ is the closure of the image of e_Ψ
- $|\Psi>$ is separating with respect to a von Neumann algebra M
  $\Longleftrightarrow$ $\ker e_\Psi = 0$ (injection)
- If the physical content of the Reeh-Schlieder theorem is relevant for the bulk, the bulk reconstruction needs to be understood in infinite-dimensional Hilbert spaces.

Von Neumann algebra in QFT

- Assume there is a unique ground state $|\Omega > \in \mathscr{H}$
- The closure of the set of states obtained by acting on $|\Omega >$ (with sums of products of smeared operator)

$\Downarrow$ *Define*

The vacuum superselection sector $\mathscr{H}_0$

- Each superselection sector of the theory is invariant subspace of the algebra of local operators

Von Neumann algebra in QFT

Open region of spacetime $u \xrightarrow{\text{Define}} A(u)$ An associated local operator algebra

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $\quad u \xrightarrow{\textit{Define}} A(u) \quad$ An associated local operator algebra

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $\quad u \xrightarrow{\textit{Define}} A(u) \quad$ An associated local operator algebra

Polar decomposition

A partial isometry

A self-adjoint Positive operator

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $\mathcal{U} \xrightarrow{\textit{Define}} A(\mathcal{U})$ An associated local operator algebra

Polar decomposition

A partial isometry

A self-adjoint Positive operator

Canonically associated with a set of projections (Spectral theorem)

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $u \xrightarrow{\textit{Define}} A(u)$ An associated local operator algebra

Polar decomposition

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A self-adjoint Positive operator

Canonically associated with a set of projections (Spectral theorem)

Von Neumann algebra $M(u) \subset B(\mathscr{H})$

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $\quad u \xrightarrow{\textit{Define}} A(u) \quad$ An associated local operator algebra

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Canonically associated with a set of projections (Spectral theorem)

Von Neumann algebra $M(u) \subset B(\mathscr{H})$ is generated by

Partial isometries

associated with the operators in M(u)

A set of all projections

Von Neumann algebra in QFT

Unbounded!

Open region of spacetime $u \xrightarrow{\text{Define}} A(u)$ An associated local operator algebra

Polar decomposition

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Canonically associated with a set of projections (Spectral theorem)

Von Neumann algebra $M(u) \subset B(\mathscr{H})$ is generated by

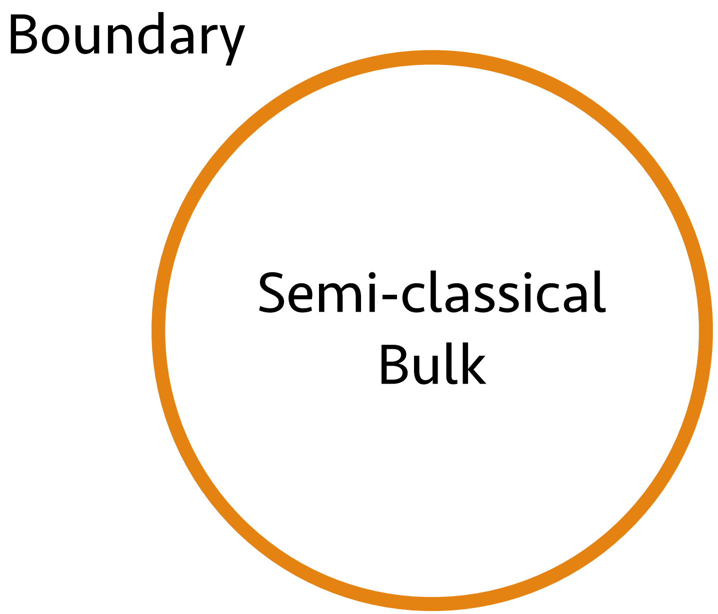
Partial isometries

associated with the operators in M(u)

A set of all projections

$\Rightarrow$ *Denote subregions in the bulk & the boundary*

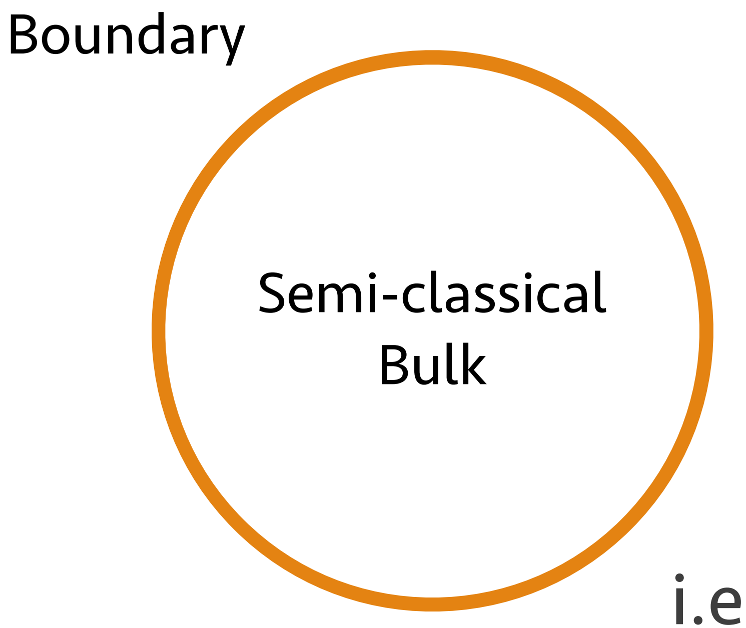
Von Neumann algebra and AdS/CFT



- Bulk theory may be effectively described by a QFT on a global AdS background
- Use von Neumann algebras to describe

  operators associated with
  covariantly defined subregions in the bulk

Von Neumann algebra and AdS/CFT



- Bulk theory may be effectively described by a QFT on a global AdS background
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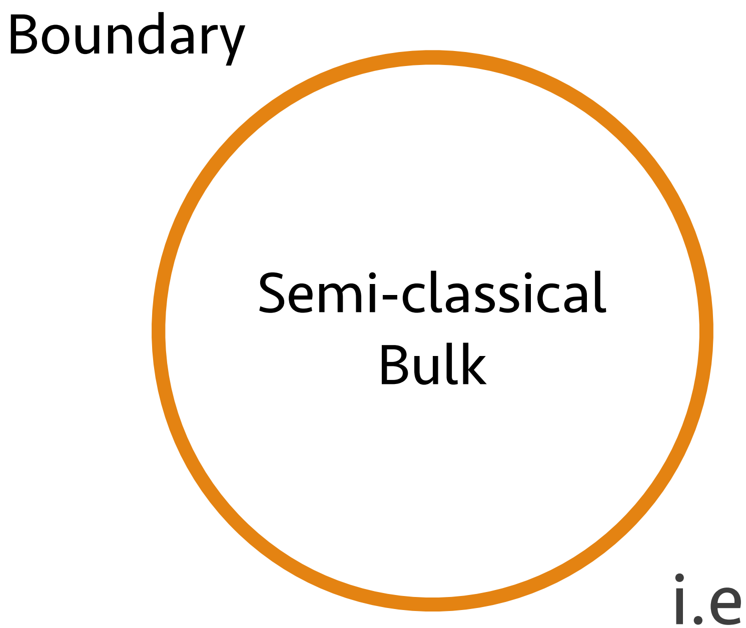
i.e. Entanglement wedge of a boundary subregion

Von Neumann algebra and AdS/CFT



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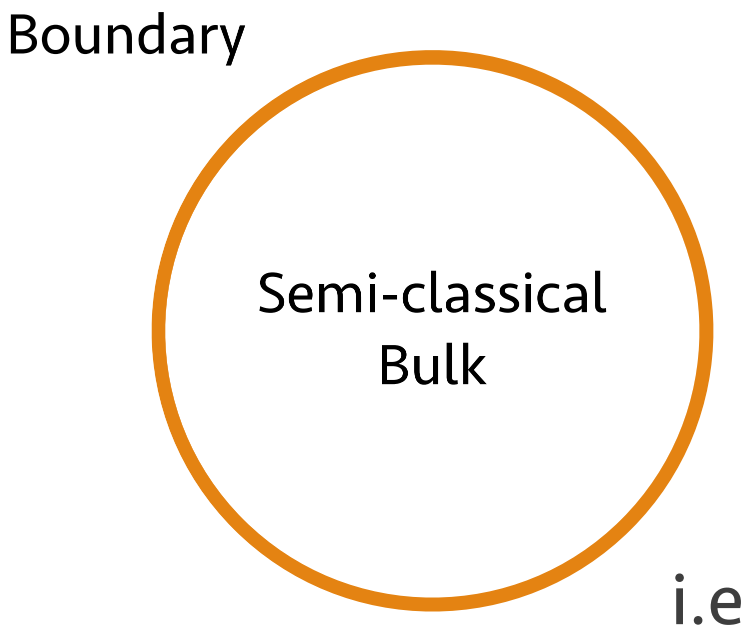
Causally complete!

Von Neumann algebra and AdS/CFT



- Bulk theory may be effectively described by a QFT on a global AdS background

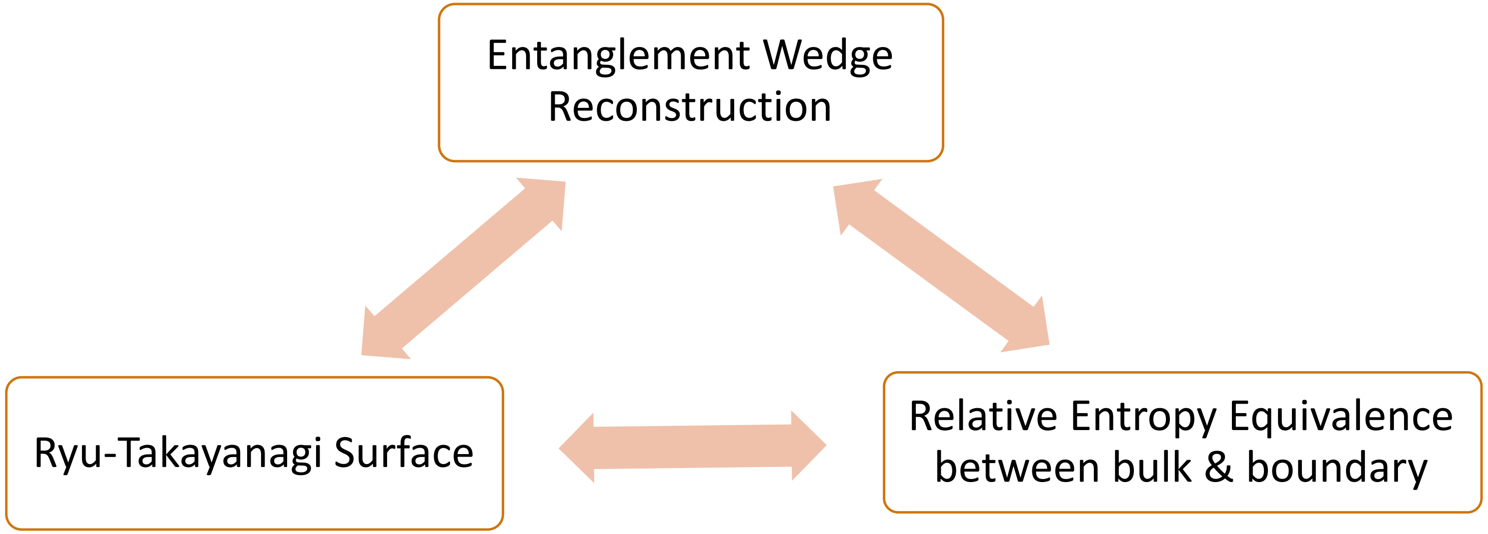
- Use von Neumann algebras to describe

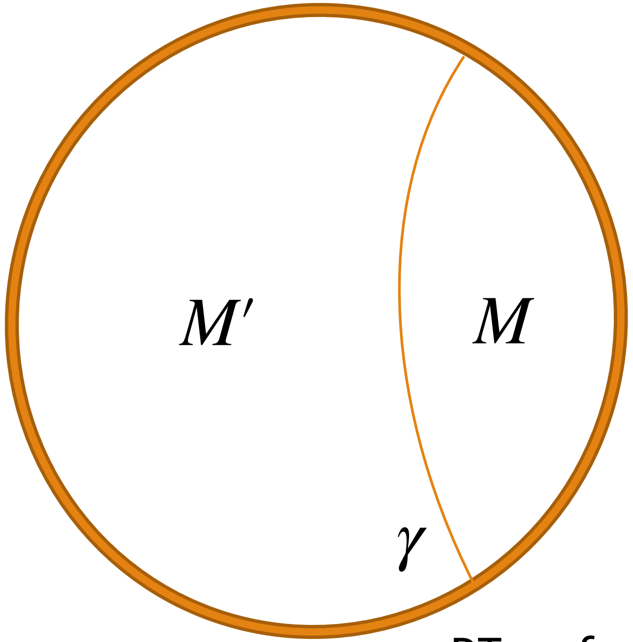
operators associated with
covariantly defined subregions in the bulk

i.e. Entanglement wedge of a boundary subregion

Causally complete! *Naturally have an associated von Neumann algebra*

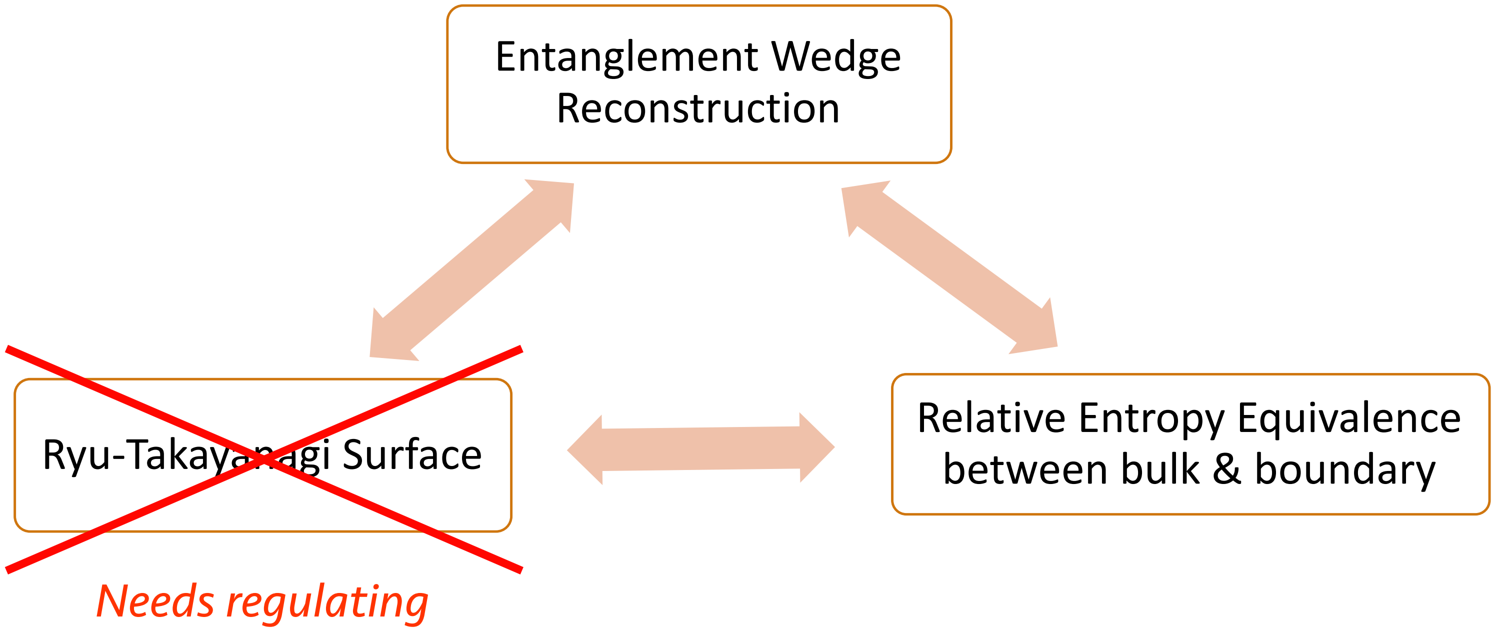
Recall:
Finite-dimensional Hilbert space



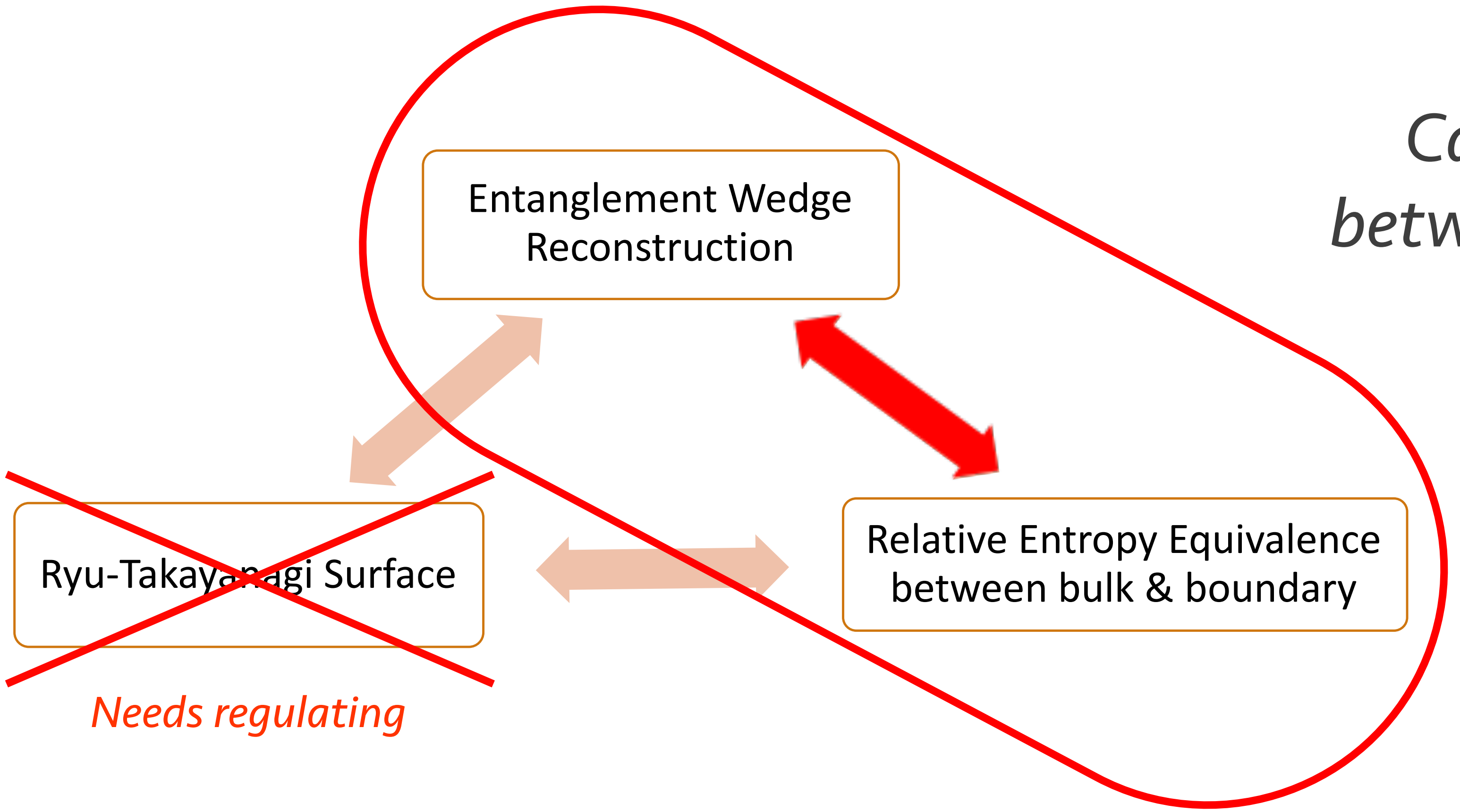


RT surface
(minimal surface)

Infinite ~~Finite~~-dimensional Hilbert space

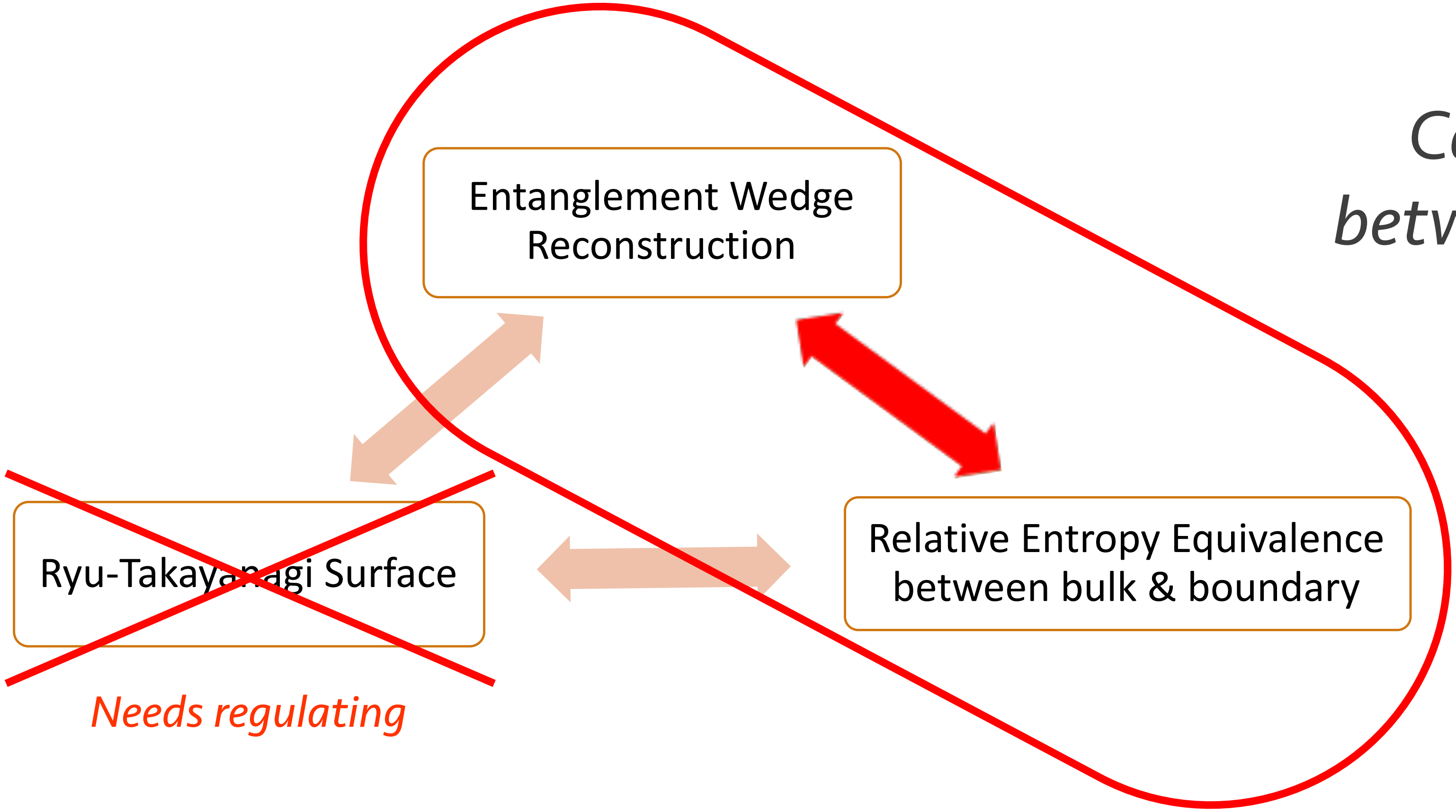


Infinite ~~Finite~~-dimensional Hilbert space



Can we have a relation between these two directly?

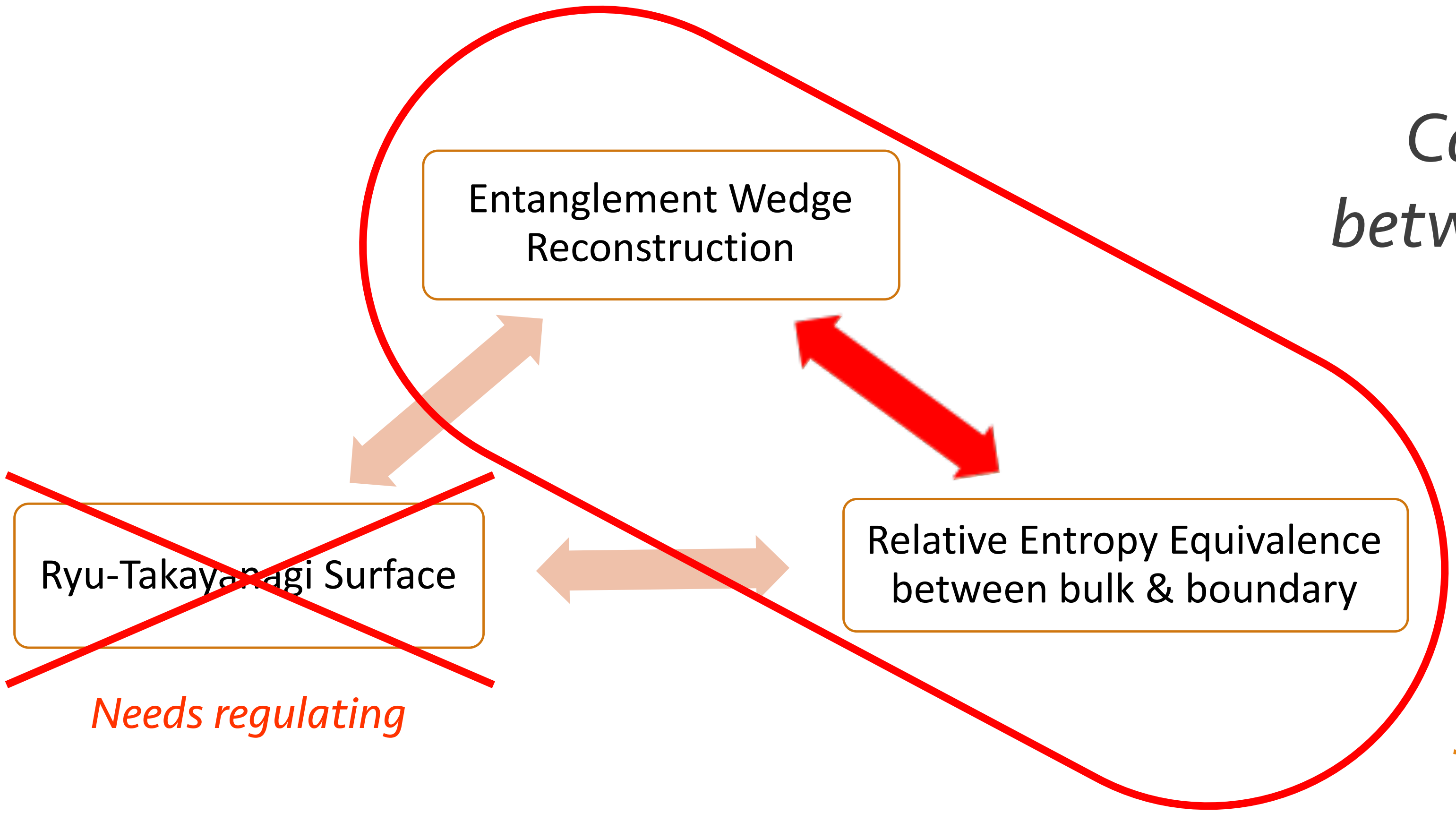
~~Finite~~ Infinite-dimensional Hilbert space



Can we have a relation between these two directly?

YES!

~~Finite~~ Infinite-dimensional Hilbert space



Can we have a relation between these two directly?

YES!

However, we need relative entropies for infinite-dimensional $\mathscr{H}$.

Relative Entropy

- ➢ In finite-dimensional Hilbert spaces: $S(\rho,\sigma) = \mathrm{Tr}\left(\rho\log\rho - \rho\log\sigma\right)$
- ➢ The relative entropy $S(\rho,\sigma)$ does not increase upon performing a partial trace on ρ and σ.
  - The relative entropy may be intuitively thought of as a measure of distinguishability between two states.
- ➢ Infinite-dimensional case needs Tomita-Takesaki theory using von Neumann algebra.

Relative Entropy using von Neumann algebra

- $|\Psi>, |\Phi> \in \mathscr{H}$, $|\Psi>$ is cyclic and separating.
- *Relative modular operator* $\Delta_{\Psi|\Phi} := \mathscr{S}^{\dagger}_{\Psi|\Phi}\mathscr{S}_{\Psi|\Phi}$

  where $\mathscr{S}_{\Psi|\Phi}$ is a *relative Tomita operator* that acts as $\mathscr{S}_{\Psi|\Phi}|x> := |y> \quad \forall \{\mathscr{O}_n\} \in M$ such that the limits $|x> = \lim_{n->\infty} \mathscr{O}_n|\Psi>$ and $|y> = \lim_{n->\infty} \mathscr{O}^{\dagger}_n|\Phi>$ exist.
- *Relative entropy* with respect to M of $|\Psi>$

$$S_{\Psi|\Phi}(M) = -<\Psi|\log\Delta_{\Psi|\Phi}|\Psi>$$

The equivalence theorem

1811.05482 (MJK, Kolchmeyer)

- For (infinite-dimensional) Hilbert spaces:

Entanglement Wedge Reconstruction ⟺ Relative Entropy Equivalence Between the bulk and the boundary

- Ingredients of the theorem:
  - An isometry $u : \mathscr{H}_{code} \to \mathscr{H}_{phys}$
  - Von Neumann algebras on $\mathscr{H}_{code}$ and $\mathscr{H}_{phys}$: $M_{code}, M'_{code}, M_{phys}, M'_{phys}$
- Assumption required:
  - If $|\Psi> \in \mathscr{H}_{code}$ is cyclic and separating with respect to M_{code}, then $u|\Psi>$ is cyclic and separating with respect to M_{phys}.

The equivalence theorem

1811.05482 (MJK, Kolchmeyer)

Entanglement Wedge Reconstruction

$\forall \mathcal{O} \in M_{code}\ \forall \mathcal{O}' \in M'_{code}$, $\exists \tilde{\mathcal{O}} \in M_{phys}\ \exists \tilde{\mathcal{O}}' \in M'_{phys}$ such that

$$\forall |\Theta> \in \mathcal{H}_{code} \quad \begin{cases} u\mathcal{O}|\Theta> = \tilde{\mathcal{O}}u|\Theta>, & u\mathcal{O}'|\Theta> = \tilde{\mathcal{O}}'u|\Theta>, \\ u\mathcal{O}^\dagger|\Theta> = \tilde{\mathcal{O}}^\dagger u|\Theta>, & u\mathcal{O}'^\dagger|\Theta> = \tilde{\mathcal{O}}'^\dagger u|\Theta>. \end{cases}$$

⇕

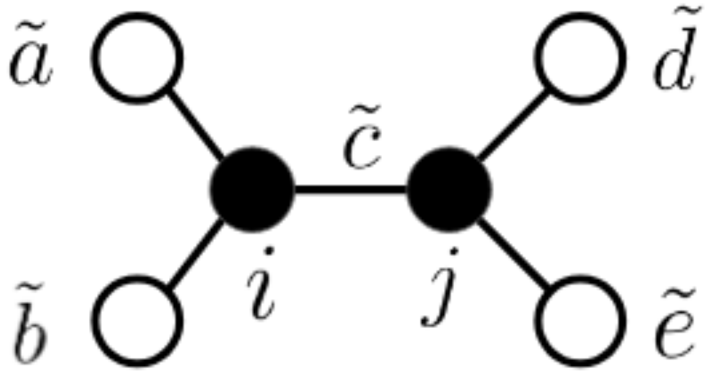
Relative Entropy Equivalence
Between the bulk and the boundary

$\forall |\Psi>, |\Phi> \in \mathcal{H}_{code}$ with $|\Psi>$ cyclic and separating w.r.t. M_{code}

$S_{\Psi|\Phi}(M_{code}) = S_{u\Psi|u\Phi}(M_{phys})$, and $S_{\Psi|\Phi}(M'_{code}) = S_{u\Psi|u\Phi}(M'_{phys})$

A toy model: tensor network

- Goal: build an explicit quantum error correcting code that is of infinite-dimensional von Neumann algebra of type II_1.
- Want a uniform tensor network — consider *qutrits*!
- Finite-dimensional collection:



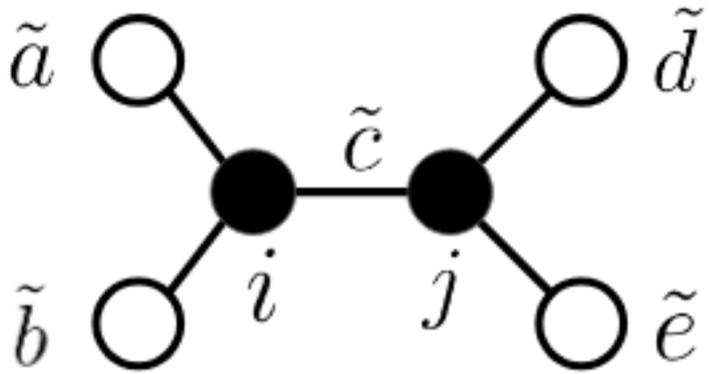
A finite-dimensional tensor network

1910.06328 (MJK, Kolchmeyer)

- ➢ Three-qutrit code

$$\begin{cases} |0> \longrightarrow \frac{1}{\sqrt{3}}\left(|\tilde{0}\tilde{0}\tilde{0}> + |\tilde{1}\tilde{1}\tilde{1}> + |\tilde{2}\tilde{2}\tilde{2}>\right), \\ |1> \longrightarrow \frac{1}{\sqrt{3}}\left(|\tilde{0}\tilde{1}\tilde{2}> + |\tilde{1}\tilde{2}\tilde{0}> + |\tilde{2}\tilde{0}\tilde{1}>\right), \\ |2> \longrightarrow \frac{1}{\sqrt{3}}\left(|\tilde{0}\tilde{2}\tilde{1}> + |\tilde{1}\tilde{0}\tilde{2}> + |\tilde{2}\tilde{1}\tilde{0}>\right). \end{cases}$$

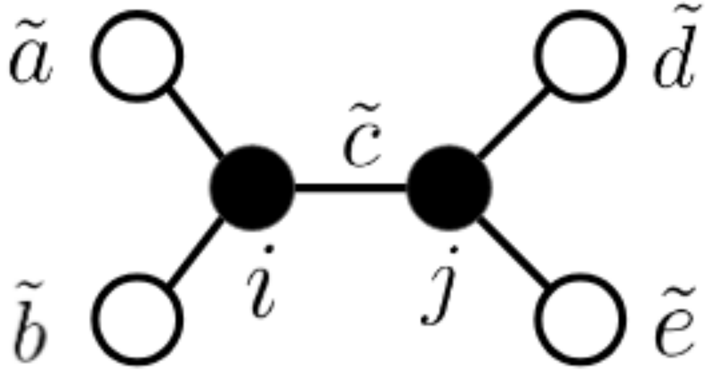
$$\Rightarrow \quad |i> \longrightarrow \sum_{\tilde{a},\tilde{b},\tilde{c}} T_{i\tilde{a}\tilde{b}\tilde{c}} |\tilde{a}\tilde{b}\tilde{c}>,$$



~ *denotes qutrits in the physical Hilbert space*

- ➢ The isometry

$$|p>_i |q>_j \longrightarrow \sum_{\tilde{x},\tilde{y},\tilde{z},\tilde{c},\tilde{w}} \sqrt{3}\, T_{p\tilde{x}\tilde{y}\tilde{c}} T_{q\tilde{z}\tilde{w}\tilde{c}} |\tilde{x}>_{\tilde{a}} |\tilde{y}>_{\tilde{b}} |\tilde{z}>_{\tilde{d}} |\tilde{w}>_{\tilde{e}}$$



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➢ Three-qutrit code

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$$\Rightarrow \quad |i> \longrightarrow \sum_{\tilde{a},\tilde{b},\tilde{c}} T_{i\tilde{a}\tilde{b}\tilde{c}} |\tilde{a}\tilde{b}\tilde{c}> ,$$

➢ Unitaries acting on a two-qutrit state

$$\begin{array}{lll} U|00> = |00> & U|11> = |20> & U|22> = |10> \\ U|01> = |11> & U|12> = |01> & U|20> = |21> \\ U|02> = |22> & U|10> = |12> & U|21> = |02> \end{array}$$

➢ The reference state

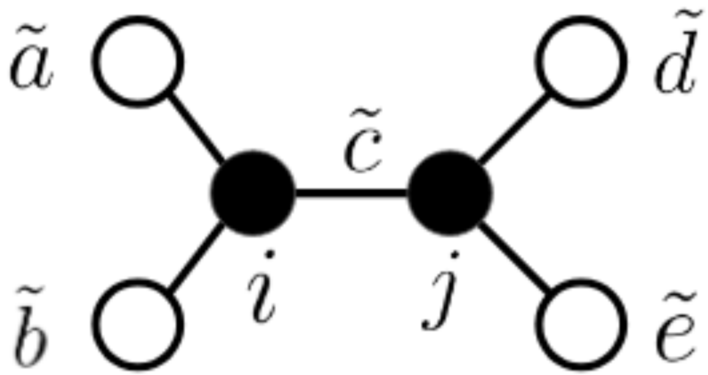
$$|\lambda> := \frac{1}{\sqrt{3}}\left[|00> + |11> + |22>\right]$$

Maximally entangled state

➢ The isometry

$$|p>_i |q>_j \quad \longrightarrow \sum_{\tilde{x},\tilde{y},\tilde{z},\tilde{c},\tilde{w}} \sqrt{3} T_{p\tilde{x}\tilde{y}\tilde{c}} T_{q\tilde{z}\tilde{w}\tilde{c}} |\tilde{x}>_{\tilde{a}} |\tilde{y}>_{\tilde{b}} |\tilde{z}>_{\tilde{d}} |\tilde{w}>_{\tilde{e}}$$

- $|\psi>_{ij}$ a vector in the Hilbert space of the black qutrits i, j
- $|\tilde{\psi}>_{\tilde{a}\tilde{b}\tilde{d}\tilde{e}}$ its image under the isometry
- $U_{\tilde{a}\tilde{b}}$ and $U_{\tilde{d}\tilde{e}}$ the unitary operator acting on qutrits $\tilde{a}, \tilde{b}$ and $\tilde{d}, \tilde{e}$
- Then: $U^{\dagger}_{\tilde{a}\tilde{b}} U^{\dagger}_{\tilde{d}\tilde{e}} |\tilde{\psi}>_{\tilde{a}\tilde{b}\tilde{d}\tilde{e}} = |\psi>_{\tilde{a}\tilde{d}} \; |\lambda>_{\tilde{b}\tilde{e}}$



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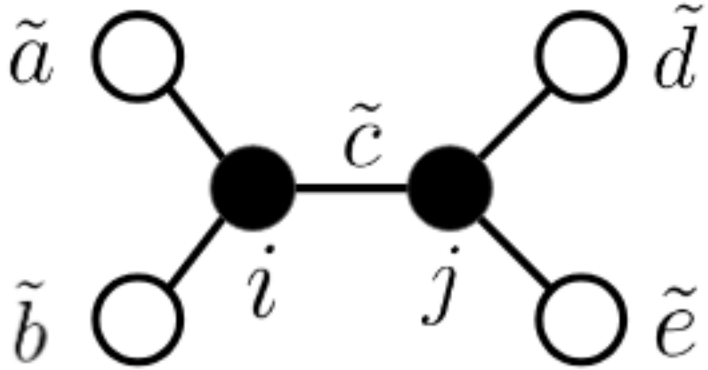
The same state as $|\psi>_{ij}$ except on white qutrits $\tilde{a}, \tilde{d}$



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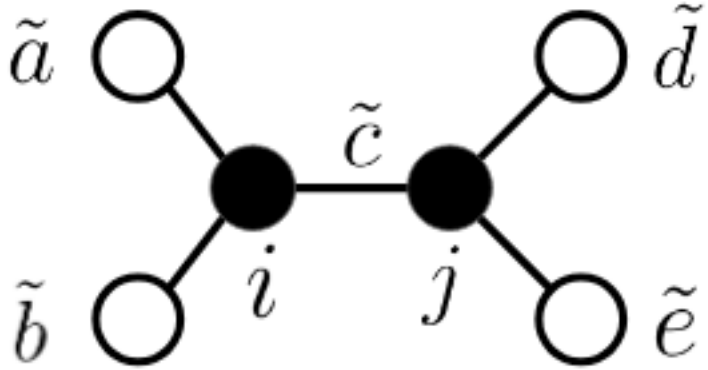
The same state as $|\psi>_{ij}$ *except on white qutrits* $\tilde{a}, \tilde{d}$

Recover

Recover

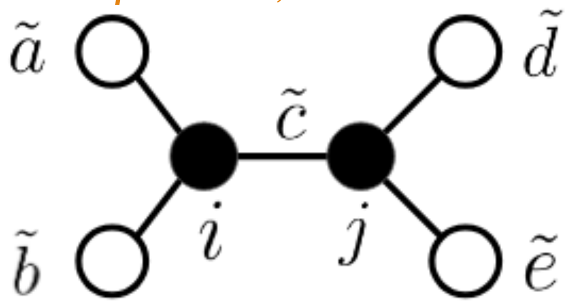


- $|\psi>_{ij}$ a vector in the Hilbert space of the black qutrits i, j
- $|\tilde{\psi}>_{\tilde{a}\tilde{b}\tilde{d}\tilde{e}}$ its image under the isometry
- $U_{\tilde{a}\tilde{b}}$ and $U_{\tilde{d}\tilde{e}}$ the unitary operator acting on qutrits $\tilde{a}, \tilde{b}$ and $\tilde{d}, \tilde{e}$
- Then: $U^{\dagger}_{\tilde{a}\tilde{b}} U^{\dagger}_{\tilde{d}\tilde{e}} |\tilde{\psi}>_{\tilde{a}\tilde{b}\tilde{d}\tilde{e}} = |\psi>_{\tilde{a}\tilde{d}} |\lambda>_{\tilde{b}\tilde{e}}$

The same state as $|\psi>_{ij}$ *except on white qutrits* $\tilde{a}, \tilde{d}$

Recover

Recover



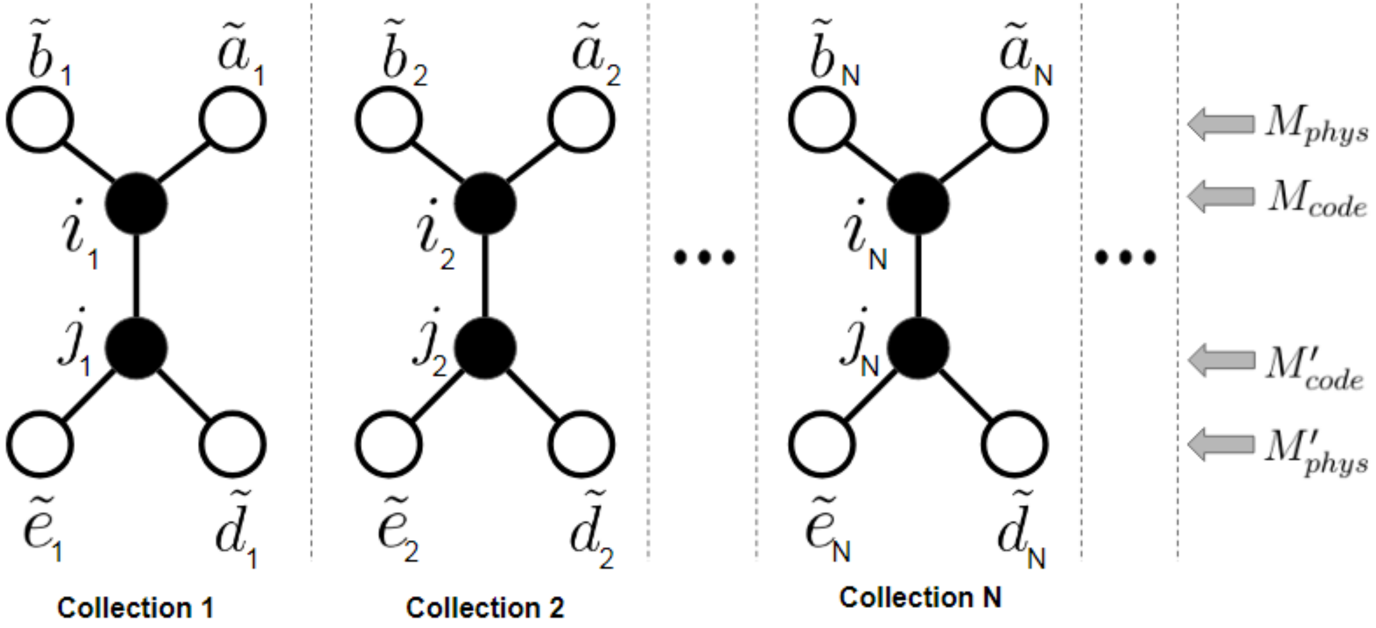
- $\mathcal{O}$ an operator that acts on the qutrit i
- $\tilde{\mathcal{O}}$ an operator that acts on the adjacent white qutrits $\tilde{a}, \tilde{b}$

$$\tilde{\mathcal{O}} := \sum_{p,q} <p|\mathcal{O}|q>_i \left[U_{\tilde{a}\tilde{b}} |p>_{\tilde{a}} <q|_{\tilde{a}} U^{\dagger}_{\tilde{a}\tilde{b}} \otimes I_{\tilde{d}\tilde{e}}\right]$$

An infinite-dimensional tensor network

1910.06328 (MJK, Kolchmeyer)

- Now juxtapose infinitely…



Construct Hilbert spaces

- Pre-Hilbert space $p\mathscr{H}_{code}$ is defined to include *states of black qutrits* where all but finitely many pairs of black qutrits are in the state $|\lambda>$.
- Any vector in $p\mathscr{H}_{code}$ = a finite linear combination of vectors in an over complete basis
- Each basis vector:

$$|M, p_1, \cdots, p_M, q_1, \cdots, q_M> := \Big[|p_1>_{i_1} |q_1>_{j1} \Big] \otimes \cdots \Big[|p_1>_{i_M} |q_1>_{jM} \Big] \otimes |\lambda>$$

Construct Hilbert spaces

- Pre-Hilbert space $p\mathscr{H}_{code}$ is defined to include *states of black qutrits* where all but finitely many pairs of black qutrits are in the state $|\lambda>$.
- Any vector in $p\mathscr{H}_{code}$ = a finite linear combination of vectors in an over complete basis
- Each basis vector: *(Renamed for convenience)*

p_k, q_k index is valued in $\{0,1,2\}$ and specifies an orthonormal basis of a black qutrit

$$|M,\{p,q\}>:=\Big[|p_1>_{i_1}|q_1>_{j1}\Big]\otimes\cdots\Big[|p_1>_{i_M}|q_1>_{jM}\Big]\otimes|\lambda>$$

Not linearly independent!

Construct Hilbert spaces

- ➢ Pre-Hilbert space $p\mathscr{H}_{code}$ is defined to include *states of black qutrits* where all but finitely many pairs of black qutrits are in the state $|\lambda>$.

- ➢ Any vector in $p\mathscr{H}_{code}$ = a finite linear combination of vectors in an over complete basis

- ➢ Each basis vector: *Not linearly independent!*

p_k, q_k index is valued in $\{0,1,2\}$ and specifies an orthonormal basis of a black qutrit

$$|M,\{p,q\}>:=\left[|p_1>_{i_1}|q_1>_{j1}\right]\otimes\cdots\left[|p_1>_{i_M}|q_1>_{jM}\right]\otimes|\lambda>$$

- ➢ Consider two basis vectors $|M,\{p,q\}_1>$ and $|M,\{p,q\}_2>$

Inner product: ignore collections beyond $\max(M_1,M_2)$

Take the usual inner product on the remaining $9^{\max(M_1,M_2)}$-dimensional Hilbert space

Construct Hilbert spaces

➢ Pre-Hilbert space $p\mathscr{H}_{code}$ is defined to include *states of black qutrits* where all but finitely many pairs of black qutrits are in the state $|\lambda>$.

➢ Any vector in $p\mathscr{H}_{code}$ = a finite linear combination of vectors in an over complete basis

➢ Each basis vector: *Not linearly independent!*

p_k, q_k index is valued in $\{0,1,2\}$ and specifies an orthonormal basis of a black qutrit

$$|M,\{p,q\}>:=\left[|p_1>_{i_1}|q_1>_{j1}\right]\otimes\cdots\left[|p_1>_{i_M}|q_1>_{jM}\right]\otimes|\lambda>$$

➢ Consider two basis vectors $|M,\{p,q\}_1>$ and $|M,\{p,q\}_2>$

Not mutually orthogonal but all normalized

Inner product: ignore collections beyond $\max(M_1, M_2)$

Take the usual inner product on the remaining $9^{\max(M_1,M_2)}$-dimensional Hilbert space

Construct Hilbert spaces

- Pre-Hilbert space $p\mathscr{H}_{code}$ is defined to include *states of black qutrits* where all but finitely many pairs of black qutrits are in the state $|\lambda>$.
- Any vector in $p\mathscr{H}_{code}$ = a finite linear combination of vectors in an over complete basis
- Each basis vector: *Not linearly independent!*

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Not mutually orthogonal but all normalized

With inner product: we can define Cauchy sequences

$\mathscr{H}_{code}$ = the closure of $p\mathscr{H}_{code}$ so that all Cauchy sequence in $\mathscr{H}_{code}$ converges

The operator algebra: closed under the strong limit

- Analogously for operators

  We can define *-algebra of operators acting on finite number of qutrits

  To get the von Neumann algebra M, we need to compute the M''

- Unlike C*-algebra, von Neumann algebra is closed under the strong limit (i.e. $\lim_{n\to\infty} \mathcal{O}_n|\Psi> = \mathcal{O}|\Psi> \ \ \forall\psi \in \mathcal{H}$)

Physical pre-Hilbert and Hilbert spaces

- Can be done similarly to construct $p\mathscr{H}_{phys}$ and $\mathscr{H}_{phys}$
- For each collection, 4 white qutrits
- Physical reference state $|\lambda\lambda>:=|\lambda>_{\tilde{a}\tilde{d}}|\lambda>_{\tilde{b}\tilde{e}}$ *Image of* $|\lambda>_{ij}$
- Construct the von Neumann algebras for the boundary

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Using in this manner, we can explicitly write down operators, von Neumann algebras as their operator algebras, unitaries, the isometry map

➡ *Showed that Entanglement Wedge reconstruction is satisfied for this toy model*

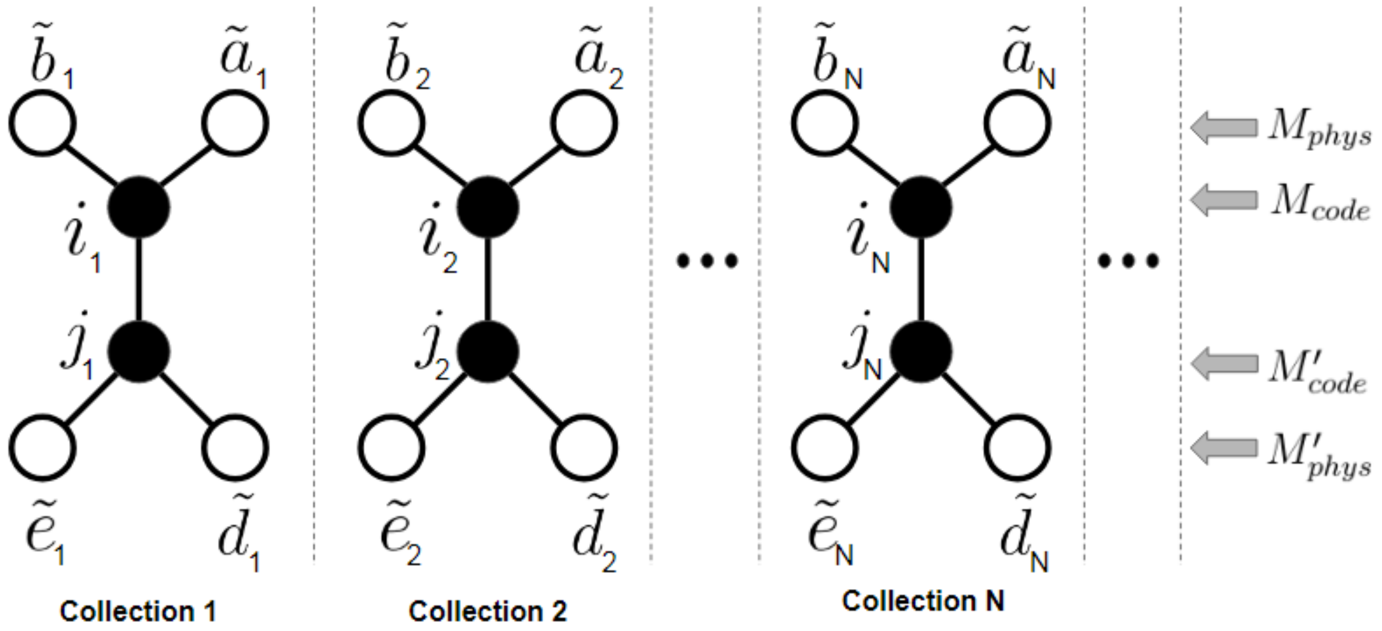
Generalize: von Neumann algebras of various type

- Previously: every collection has a reference state $|\lambda>:=\frac{1}{\sqrt{3}}\left[|00>+|11>+|22>\right]$
- An infinite sequence of (separable) Hilbert spaces $\mathscr{H}_n$, each equipped with a reference state $|\lambda_n>:=\frac{1}{\sqrt{1+\alpha^2+\beta^2}}\left[|00>+\alpha|11>+\beta|22>\right]$

To appear (MJK, Tang)

- $\alpha=\beta=0$: type I_∞
- $\alpha=\beta=1$: type II_1 (the previous case, maximally entangled state)
- $\alpha=1$ and $\beta=0$: type II_∞
- $\alpha,\beta\neq 0\ and\ \log\alpha/\log\beta\notin\mathbb{Q}$: type III_1 (the generic operator algebra of local QFTs)
- $\alpha=\gamma^k,\ \beta=\gamma^\ell\ \ for\ k,\ell\in\mathbb{Z}_+$ and $0<\gamma<1$: type III_λ for $\lambda=\gamma^{\gcd(k,\ell)}$
- ($\alpha<1$ and $\beta=0$) or ($\alpha=0$ and $\beta<1$) : type III_α or III_β
- $\alpha>1$ and $\beta=0$ or $\alpha=0$ and $\beta>1$: type $III_{\alpha^{-1}}$ or $III_{\beta^{-1}}$

Tensor product of types of vN algebras

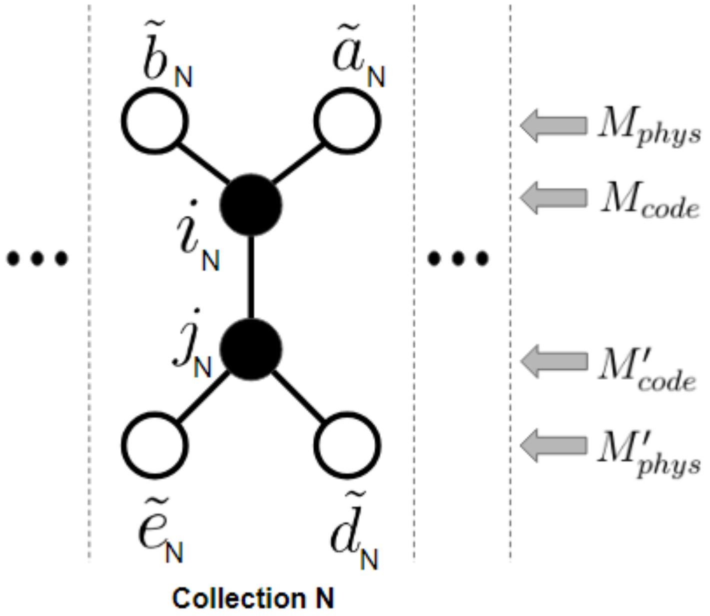


- More generally putting: type $T \to$ type $T \times II_1 =$ type?

Tensor product of types of vN algebras

To appear (MJK, Tang)

Generalize for $M_A \otimes M_B$



type $T \to$ type $T \times II_1 =$ type?

$A\backslash B$	I_n	I_∞	II_1	II_∞	III_λ	III_1
I_m	I_{nm}	I_∞	II_1	II_∞	III_λ	III_1
I_∞	I_∞	I_∞	II_∞	II_∞	III_λ	III_1
II_1	II_1	II_∞	II_1	II_∞	III_λ	III_1
II_∞	II_∞	II_∞	II_∞	II_∞	III_λ	III_1
III_μ	III_μ	III_μ	III_μ	III_μ	III_σ	III_1
III_1	III_1	III_1	III_1	III_1	III_1	III_1

$$0 < \mu, \lambda < 1, \qquad \sigma = \begin{cases} \alpha^{\gcd(k,\ell)} & if \;\; \log\lambda / \log\mu \in \mathbb{Q} \\ 1 & otherwise \end{cases}$$

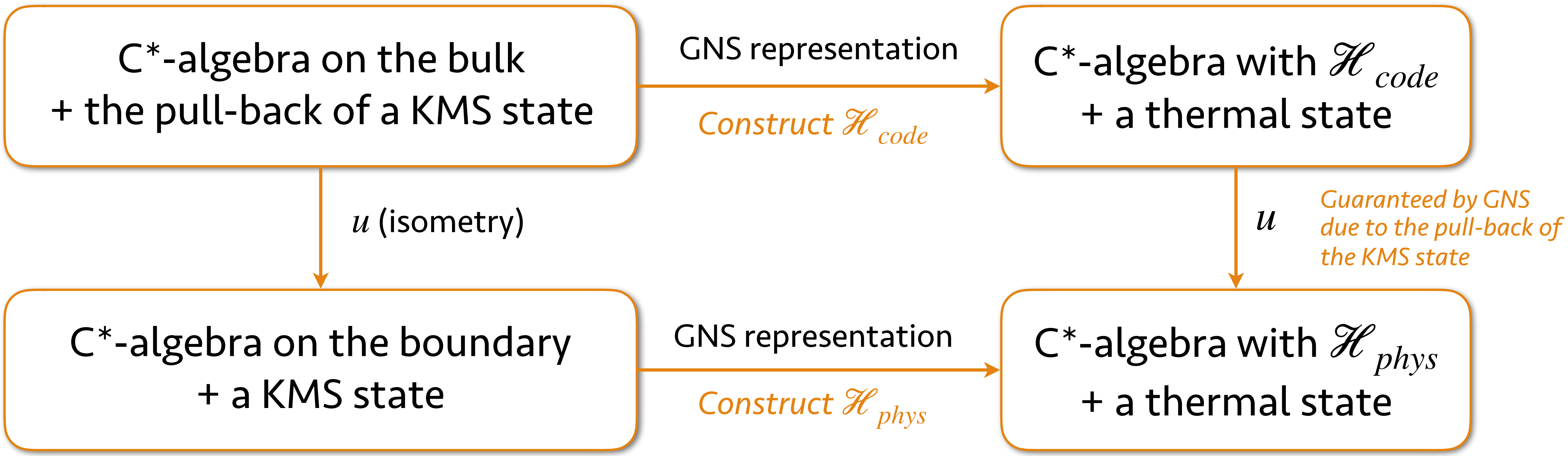
More complex QECC

- For more complicated quantum error correcting codes (cf. HaPPY code)
  - No way to construct the Hilbert space directly due to high complexity.
-

The toy model considered	State-pushing
More complex QECC	Operator-pushing

- Now easier to work with C*-algebra instead of von Neumann algebra as the first step and connect to von Neumann algebra afterwards for (thermal) states and relative entropies.
- Von Neumann algebra is state-dependent but C*-algebra is not (with no 'commutant' either)

To appear (MJK, Gesteau)



- The thermal state: cyclic & separating on C*-algebra and von Neumann algebra
- We can extend to more nontrivial QECCs and have entanglement wedge reconstructions (cf. HaPPY code)

Thank you for listening!