Problem 1 (20 points)

a) (8 points)

$$f(x) = \begin{array}{ll} 0 & -\pi < x \leq 0 \\ x & 0 \leq x < \pi \end{array} \tag{1}$$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n\, e^{i\,n\,x} \tag{2}$$

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, e^{-i\,n\,y}\, dy = \frac{1}{2\pi} \int_{0}^{\pi} y\, e^{-i\,n\,y}\, dy = \begin{array}{ll} \frac{e^{-i\,n\,\pi}\,(1+i\,n\,\pi)-1}{2\,\pi\,n^2} & n \neq 0 \\ \pi/4 & n = 0 \end{array}$$

b) (6 points)

$$g(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\left(\frac{e^{-i\,n\,\pi}\,(1+i\,n\,\pi)-1}{2\,\pi\,n^2}\right) \mathrm{Sin}(n\,x) \tag{3}$$

$$h(x) = \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{e^{-i\,n\,\pi}\,(1+i\,n\,\pi)-1}{2\,\pi\,n^2}\right) \mathrm{Cos}(n\,x)$$

The sine terms, g(x), should represent the odd extension of x/2 from [0,π] to the whole real line:

$$g_{\mathrm{oddextension}} = (x - 2\,k\,\pi)/2 \quad \text{for} \quad (2\,k-1)\,\pi < x < (2\,k+1)\,\pi \tag{4}$$

By simplifying the series g(x) we get

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\, \mathrm{Sin}(n\,x) \tag{5}$$
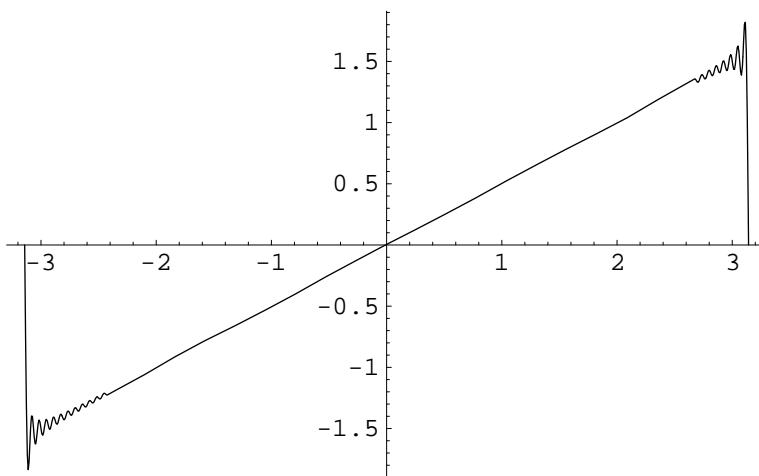
A simple plot of 100 terms confirms our suspicion.

Figure 1

The cosine terms, h(x), should represent the even extension of x/2 from [0,π] to the whole real line

$$g_{\mathrm{evenextension}} = |x - 2\,k\,\pi|/2 \quad \text{for} \quad (2\,k-1)\,\pi < x < (2\,k+1)\,\pi \tag{6}$$

By simplifying the series for h(x) we get

$$h(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \, \text{Cos}(n\,x) \tag{7}$$
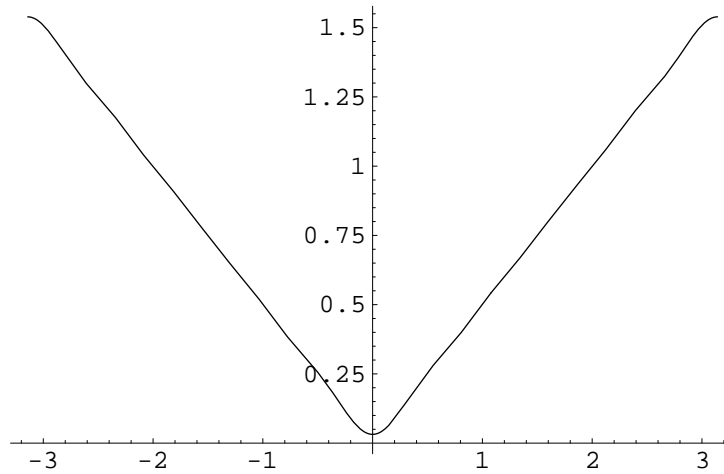
A simple plot of 10 terms confirms our suspicion.

Figure 2

c) (6points)

$$g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \, \text{Cos}(n\,x) = \begin{array}{ll} 1/2 & -\pi < x \le 0 \\ 1/2 & 0 \le x < \pi \end{array} \tag{8}$$

$$h'(x) = \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{\pi n} \, \text{Sin}(n\,x) = \begin{array}{ll} -1/2 & -\pi < x \le 0 \\ 1/2 & 0 \le x < \pi \end{array} \tag{9}$$

Since the function that g(x) is the fourier sine series of is continuously differentiable on $(-\pi,\pi)$ with slope 1/2, we expect that the series found by differentiating the sine series term by term should be the fourier series of 1/2. However we are wrong, as the following plot of 100 terms shows
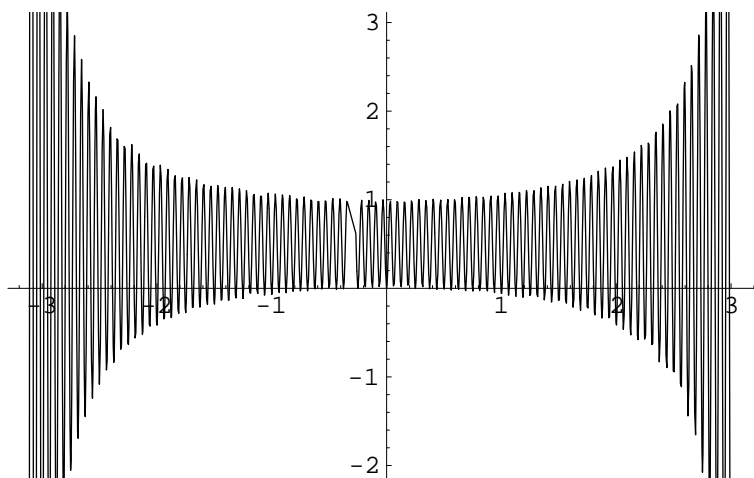

Figure 3

The problem is that the series g(x) is slowly convergent, it fails the Weierstrass M-Test for all x, so probably isn't uniformly convergent anywhere. **(Give bonus points for proving non-uniform convergence)** As a result, the term by term derivative of g(x), whose coefficients don't decay, doesn't converge anywhere. The reason for the slow convergence of g(x) is that the odd periodic extension of g(x) is a discontinuous function, i.e. it jumps by 2π at $x=(2k+)\pi$ for all integers k. The series found from term by term differentiation of g(x) is attempting to approximate a function which is the derivative

of a discontinuous function, i.e. g'(x) represents 1/2 plus a bunch of delta functions centered at x=(2k+1)π.

The differentiated series still "converges" to 1/2 plus a bunch of delta functions in the sense discussed in the appendix of problem set 5 solutions.

Since h(x) is the cosine series representation of a function which is not differentiable at x=0, we expect that the series found by differentiating the cosine part term by term should be a step function. A plot of 100 terms confirms this
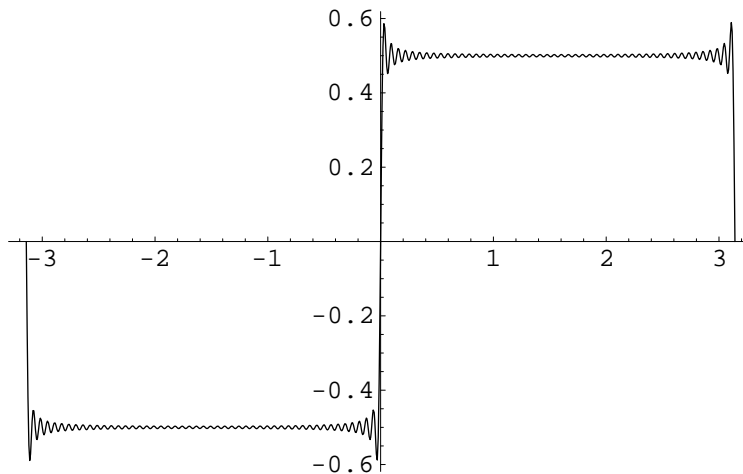

Figure 4

Notice that the series h(x) is uniformly convergent by the Weierstrass M-Test

$$\left| \frac{(-1)^n - 1}{\pi n^2} \, \text{Cos}\,(n\,x) \right| \le \left| \frac{(-1)^n - 1}{\pi n^2} \right| \le \frac{2}{\pi n^2} \tag{10}$$

and

$$\sum_{n=1}^{\infty} \frac{2}{\pi n^2} \tag{11}$$

converges (to π/3). The term by term derivative of h(x) doesn't pass the M-Test and so likely isn't uniformly convergent. **(give bonus points for proving non-uniform convergence)** So we shouldn't expect the derivative to be a continuous function everywhere, and indeed it isn't at x=0. Compare the convergence properties to that of the function g(x). Here, the even periodic extension is a continuous function, leading to more rapid convergence of both h(x) and it's termwise derivative.

Problem 2 (2×7 points)

a)

$$e^{-a^2\,x^2} \tag{12}$$

By completing the square we have:

$$\mathcal{F}\left(e^{-a^2\,x^2}\right) = \int_{-\infty}^{\infty} e^{-2\pi i f x}\, e^{-a^2\,x^2}\, dx = e^{-\left(\frac{\pi f}{a}\right)^2} \int_{-\infty}^{\infty} e^{-a^2\left(x + \frac{\pi i f}{a^2}\right)^2}\, dx \tag{13}$$

Make the change of variable

$$y = a\,x + \frac{\pi i f}{a} \tag{14}$$

The integral becomes

$$\int_{-\infty}^{\infty} e^{-a^2\left(x + \frac{\pi i f}{a^2}\right)^2}\, dx = \frac{1}{a} \int_{-\infty + \frac{\pi i f}{a}}^{\infty + \frac{\pi i f}{a}} e^{-y^2}\, dy \tag{15}$$

Since the integrand is an entire function, we can deform this contour continuously in the complex plane into another contour more easy to evaluate without changing the value of the integral.

$$\int_{-\infty+\frac{\pi i f}{a}}^{\infty+\frac{\pi i f}{a}} e^{-y^2} \, dy = \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi} \tag{16}$$

So we conclude

$$\mathcal{F}\left(e^{-a^2 x^2}\right) = \frac{\sqrt{\pi}}{a} e^{-\left(\frac{\pi f}{a}\right)^2} \tag{17}$$

b)

$$\mathcal{F}\left(\frac{1}{x}\right) = \mathrm{PV} \int_{-\infty}^{\infty} \frac{e^{-2\pi i f x}}{x} \, dx \tag{18}$$

For f<0 use the following contour which we call $\Gamma_{R,\epsilon}$
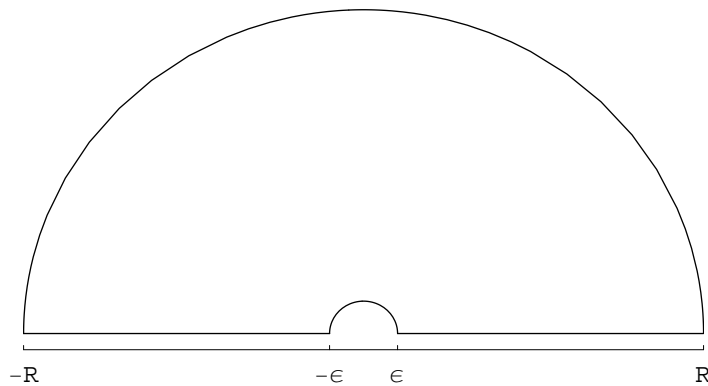

Figure 5

Since the integrand is analytic on and inside $\Gamma_{R,\epsilon}$ the Cauchy-Goursat theorem tells us that this integral will vanish. A counter-clockwise traversal of this contour gives

$$0 = \int_{\Gamma} \frac{e^{-2\pi i f z}}{z} \, dz = \int_{\epsilon}^{R} \frac{e^{-2\pi i f z}}{z} \, dz + \int_{C_R} \frac{e^{-2\pi i f z}}{z} \, dz + \int_{-R}^{-\epsilon} \frac{e^{-2\pi i f z}}{z} \, dz + \int_{C_\epsilon} \frac{e^{-2\pi i f z}}{z} \, dz \tag{19}$$

We now evaluate/bound each portion of this. The integral around the large circular contour vanishes by Jordan's lemma. The details are shown here:

$$\left| \int_{C_R} \frac{e^{-2\pi i f z}}{z} \, dz \right| = \left| \int_0^\pi \frac{e^{-2\pi i f R e^{i\theta}}}{R e^{i\theta}} R i e^{i\theta} \, d\theta \right| \leq \int_0^\pi \left| e^{-2\pi i f R e^{i\theta}} \right| \, d\theta = \int_0^\pi e^{2\pi f R \sin\theta} \, d\theta \tag{20}$$

Jordan's lemma tells us that since f<0

$$f \sin\theta \leq \frac{2f}{\pi} \theta \quad \text{on } [0, \pi/2]$$

$$f \sin\theta \leq \frac{2f}{\pi} (\pi - \theta) \quad \text{on } [\pi/2, \pi] \tag{21}$$

These give us

$$\int_0^{\pi/2} e^{2\pi f R \sin\theta} \, d\theta \leq \int_0^{\pi/2} e^{4 R f \theta} \, d\theta = \frac{e^{2 R f \pi} - 1}{4 R f} \to 0$$

$$\int_{\pi/2}^{\pi} e^{2\pi f R \sin\theta} \, d\theta \leq \int_{\pi/2}^{\pi} e^{4 R f (\pi - \theta)} \, d\theta = \frac{e^{2\pi R f} - 1}{4 R f} \to 0 \tag{22}$$

So as R→∞

$$\int_{C_R} \frac{e^{-2\pi i f z}}{z} \, dz \to 0 \tag{23}$$

On the small semicircular contour we have

$$\int_{C_\epsilon} \frac{e^{-2\pi i f z}}{z} \, dz = \int_\pi^0 \frac{e^{-2\pi i f \epsilon e^{i\theta}}}{\epsilon \, e^{i\theta}} \, \epsilon \, i \, e^{i\theta} \, d\theta = \int_\pi^0 e^{-2\pi i f \epsilon e^{i\theta}} \, i \, d\theta \to \int_\pi^0 i \, d\theta = -\pi \, i \tag{24}$$

So we find for f<0

$$\mathcal{F}\left(\frac{1}{x}\right) = \mathrm{PV} \int_{-\infty}^\infty \frac{e^{-2\pi i f x}}{x} \, dx =$$

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left(\int_\epsilon^R \frac{e^{-2\pi i f z}}{z} \, dz + \int_{-R}^{-\epsilon} \frac{e^{-2\pi i f z}}{z} \, dz \right) = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left(-\int_{C_R} \frac{e^{-2\pi i f z}}{z} \, dz - \int_{C_\epsilon} \frac{e^{-2\pi i f z}}{z} \, dz \right) = \pi \, i \tag{25}$$

For f>0, notice that by a change of integration variable y=-x we have

$$\mathrm{PV} \int_{-\infty}^\infty \frac{e^{-2\pi i f x}}{x} \, dx = -\mathrm{PV} \int_{-\infty}^\infty \frac{e^{-2\pi i (-f) y}}{y} \, dy = -\pi \, i \tag{26}$$

since -f<0. So we conclude

$$\mathcal{F}\left(\frac{1}{x}\right) = \begin{array}{ll} -\pi \, i & f > 0 \\ \pi \, i & f < 0 \end{array} \tag{27}$$

Problem 3 (25 points)

a) (10 points)

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$
$$c(x, 0) = c_0 \, \delta(x - \xi)$$
$$c(\pm\infty, t) = 0 \tag{28}$$

$$\overline{c}(\omega, t) = \int_{-\infty}^\infty e^{-2\pi i f x} \, c(x, t) \, dx \tag{29}$$

Fourier transforming the left side of the equation gives

$$\mathcal{F}\left(\frac{\partial c}{\partial t}\right) = \int_{-\infty}^\infty e^{-2\pi i f x} \frac{\partial c}{\partial t} \, dx = \frac{\partial}{\partial t} \int_{-\infty}^\infty e^{-2\pi i f x} \, c \, dx = \frac{\partial \overline{c}}{\partial t} \tag{30}$$

Fourier transforming the right side of the equation and using integration by parts gives

$$\mathcal{F}\left(\frac{\partial^2 c}{\partial x^2}\right) = \int_{-\infty}^\infty e^{-2\pi i f x} \frac{\partial^2 c}{\partial x^2} \, dx = (2\pi i f)^2 \int_{-\infty}^\infty e^{-2\pi i f x} \, c \, dx = -4\pi^2 \, f^2 \, \overline{c} \tag{31}$$

In the integration by parts procedure it was assumed that c and c_x vanish at ±∞. Transforming this initial condition gives

$$\mathcal{F}(c_0 \, \delta(x - \xi)) = c_0 \int_{-\infty}^\infty e^{-2\pi i f x} \, \delta(x - \xi) \, dx = c_0 \, e^{-2\pi i f \xi} \tag{32}$$

So the transformed problem is

$$\frac{\partial \overline{c}}{\partial t} = -4 \pi^2 \, f^2 \, D \, \overline{c}$$

$$\overline{c} \, (f, \, 0) = c_0 \, e^{-2 \pi i f \xi}$$

(33

The solution (found with integrating factors or the general solution formula) is

$$\overline{c} \, (f, \, t) = c_0 \, e^{-2 \pi i f \xi} \, e^{-4 \pi^2 \, f^2 \, D t}$$

(34)

To get c(x,t) we must invert the transform

$$c \, (x, \, t) = \mathcal{F}^{-1} \, (\overline{c} \, (f, \, t)) = c_0 \int_{-\infty}^{\infty} e^{2 \pi i f (x - \xi)} \, e^{-4 \pi^2 \, f^2 \, D t} \, df$$

(35)

In problem 2a we found

$$\int_{-\infty}^{\infty} e^{-2 \pi i y z} \, e^{-a^2 \, z^2} \, dz = \frac{\sqrt{\pi}}{a} \, e^{-\left(\frac{\pi y}{a}\right)^2}$$

(36)

The integral we're evaluating is identical to this with

$$z = f$$
$$a = 2 \pi \sqrt{D t}$$
$$y = \xi - x$$

(37)

So we conclude

$$\int_{-\infty}^{\infty} e^{2 \pi i f (x - \xi)} \, e^{-4 \pi^2 \, f^2 \, t} \, dx = \frac{\sqrt{\pi}}{2 \pi \sqrt{D t}} \, e^{-\left(\frac{\pi (\xi - x)}{2 \pi \sqrt{D t}}\right)^2}$$

(38)

or after simplifying

$$c \, (x, \, t) = \frac{c_0}{\sqrt{4 \pi D t}} \, e^{-\frac{(x - \xi)^2}{4 D t}}$$

(39)

b)(10 points)

$$\frac{\partial c}{\partial t} = D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)$$

$$c \, (x, \, y, \, z, \, 0) = c_0 \, \delta \, (x - \xi) \, \delta \, (y - \eta) \, \delta \, (z - \zeta)$$

$$\lim_{x^2 + y^2 + z^2 \to \infty} c = 0$$

(40)

The Fourier transform is defined as

$$\overline{c} \, (\omega, \, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2 \pi i (f_x, f_y, f_z) \cdot (x, y, z)} \, c \, (x, \, y, \, z, \, t) \, dx \, dy \, dz$$

(41)

As in the previous part

$$\mathcal{F} \left(\frac{\partial c}{\partial t} \right) = \frac{\partial \overline{c}}{\partial t}$$

(42)

Also as in the previous part, integration by parts may be performed to calculate

$$\mathcal{F} \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) = -4 \pi^2 \, (f_x{}^2 + f_y{}^2 + f_z{}^2) \, \overline{c} =$$

(43)

The initial condition is equally simple to calculate

$$\mathcal{F} \, (c_0 \, \delta \, (x - \xi) \, \delta \, (y - \eta) \, \delta \, (z - \zeta)) = c_0 \, e^{-2 \pi i (f_x, f_y, f_z) \cdot (\xi, \eta, \zeta)}$$

(44)

So our problem becomes

$$\frac{\partial \overline{c}}{\partial t} = -4\,\pi^2\,(f_x{}^2 + f_y{}^2 + f_z{}^2)\,D\,\overline{c}$$

$$c\,(\omega,\,0) = c_0\ e^{-2\,\pi\,i\,(f_x,f_y,f_z)\cdot(\xi,\eta,\zeta)}$$

<div align="right">(45</div>

The solution to this is

$$\overline{c} = c_0\ e^{-2\,\pi\,i\,(f_x,f_y,f_z)\cdot(\xi,\eta,\zeta)}\ e^{-4\,\pi^2\,(f_x{}^2+f_y{}^2+f_z{}^2)\,D\,t}$$

<div align="right">(46)</div>

Inverting is done just as in the last part

$$c\,(x,\,y,\,z,\,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\,\pi\,i\,(f_x,f_y,f_z)\cdot(x,y,z)}\ \overline{c}\,(f_x,\,f_y,\,f_z,\,t)\,df_x\,df_y\,df_z$$

<div align="right">(47)</div>

Which, after inserting the expression for $\overline{c}$, can be written

$$c_0\left(\int_{-\infty}^{\infty} e^{2\,\pi\,i\,f_x\,x}\ e^{-2\,\pi\,i\,f_x\,\xi}\ e^{-4\,\pi^2\,f_x{}^2\,D\,t}\ df_x\right)\times$$

$$\left(\int_{-\infty}^{\infty} e^{2\,\pi\,i\,f_y\,y}\ e^{-2\,\pi\,i\,f_y\,\eta}\ e^{-4\,\pi^2\,f_y{}^2\,D\,t}\ df_y\right)\times\left(\int_{-\infty}^{\infty} e^{2\,\pi\,i\,f_z\,z}\ e^{-2\,\pi\,i\,f_z\,\zeta}\ e^{-4\,\pi^2\,f_z{}^2\,D\,t}\ df_z\right)$$

<div align="right">(48)</div>

Notice that in this form, each of the integrals is exactly like the one we calculated in the previous part, so we know that this last expression becomes

$$c_0\left(\frac{1}{\sqrt{4\,\pi\,D\,t}}\ e^{-\frac{(x-\xi)^2}{4\,D\,t}}\right)\left(\frac{1}{\sqrt{4\,\pi\,D\,t}}\ e^{-\frac{(y-\eta)^2}{4\,D\,t}}\right)\left(\frac{1}{\sqrt{4\,\pi\,D\,t}}\ e^{-\frac{(z-\zeta)^2}{4\,D\,t}}\right)$$

<div align="right">(49)</div>

Which simplifies to

$$\frac{c_0}{(4\,\pi\,D\,t)^{3/2}}\ e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}{4\,D\,t}}$$

<div align="right">(50)</div>

c)(5 points)

We suspect that by superposition the function found in the last part, call it G, can be used to solve

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

$$c\,(x,\,y,\,z,\,0) = g\,(x,\,y,\,z)$$

<div align="right">(51)</div>

Define a new function u as follows

$$u\,(x,\,y,\,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\,(\xi,\,\eta,\,\zeta)\,G\,(x,\,y,\,z,\,t\,|\,\xi,\,\eta,\,\zeta)\,d\xi\,d\eta\,d\zeta$$

<div align="right">(52)</div>

We will show that this solves the PDE and satisfies the initial condition. For shorthand, denote the triple integral over all 3-space as

$$\int dx$$

<div align="right">(53)</div>

and use vector notation

$$X = (x,\,y,\,z)$$

$$\Xi = (\xi,\,\eta,\,\zeta)$$

<div align="right">(54)</div>

so

$$u\,(X) = \int g\,(\Xi)\,G\,(X,\,t\,|\,\Xi)\,d\Xi$$

<div align="right">(55)</div>

we calculate

$$u(X, 0) = \int g(\Xi) G(X, 0 \mid \Xi) \, d\Xi = \int g(\Xi) \delta(X - \Xi) \, d\Xi = g(X) \tag{56}$$

assuming that derivatives can pass freely through the integral and recalling that $G_t = D \Delta G$

$$u_t = \int g(\Xi) G_t(X, t \mid \Xi) \, d\Xi = D \int g(\Xi) \Delta G(X, t \mid \Xi) \, d\Xi = D\Delta \int g(\Xi) G(X, t \mid \Xi) \, d\Xi = D \Delta u \tag{57}$$

So u defined this way satisfies both the initial condition and the PDE

Problem 4 (5×5 points)

$$y'' + \lambda^2 y = f(x)$$
$$y(0) = y'(1) = 0 \tag{58}$$

a)

$$y'' + \lambda^2 y = 0$$
$$y(0) = y'(1) = 0 \tag{59}$$

Since this is a constant coefficient linear ODE it will have at least one solution of the form e^{rx}. Plugging this in

$$r^2 + \lambda^2 = 0 \tag{60}$$

So the general solution is of the form

$$y = A e^{i\lambda x} + B e^{-i\lambda x} \tag{61}$$

The boundary conditions give

$$A + B = 0$$
$$A \lambda e^{i\lambda} - B \lambda e^{-i\lambda} = 0 \tag{62}$$

Systems of equations only have non-trivial solutions when the determinant of the corresponding matrix is zero

$$0 = \begin{vmatrix} 1 & 1 \\ \lambda e^{i\lambda} & -\lambda e^{-i\lambda} \end{vmatrix} = \lambda(-e^{-i\lambda} - e^{i\lambda}) \tag{63}$$

$\lambda = 0$ is one solution, other solutions of this are found by rearranging

$$e^{2i\lambda} = -1 = e^{\pi(2n+1)i} \tag{64}$$

a quick check shows that $\lambda = 0$ gives y=0 which is trivial. The other values of λ give

$$\lambda_n = (2n + 1)\pi/2$$
$$y_n = \mathrm{Sin}(\lambda_n x) \tag{65}$$

b)

$$G'' + \lambda^2 G = \delta(x - \xi)$$
$$G(0) = G'(1) = 0 \tag{66}$$

Method 1: The Usual Approach

If $\lambda \neq 0$, the Green's function will be of the form

$$G = \begin{array}{ll} A \mathrm{Sin}(\lambda x) + B \mathrm{Cos}(\lambda x) & 0 \leq x < \xi \\ C \mathrm{Sin}(\lambda x) + D \mathrm{Cos}(\lambda x) & \xi < x \leq 1 \end{array} \tag{67}$$

To determine the constants, we first fit the boundary conditions

$$B = 0$$
$$C \mathrm{Cos}(\lambda) - D \mathrm{Sin}(\lambda) = 0 \tag{68}$$

These give

$$G = \begin{array}{ll} A \, \text{Sin} \, (\lambda \, x) & 0 \le x < \xi \\ E \, \text{Cos} \, (\lambda \, (x-1)) & \xi < x \le 1 \end{array} \tag{69}$$

Continuity and the jump condition on the derivative at x=ξ require

$$\begin{aligned} A \, \text{Sin} \, (\lambda \, \xi) &= E \, \text{Cos} \, (\lambda \, (\xi - 1)) \\ - E \, \lambda \, \text{Sin} \, (\lambda \, (\xi - 1)) &- A \, \lambda \, \text{Cos} \, (\lambda \, \xi) = 1 \end{aligned} \tag{70}$$

We conclude

$$G = \begin{array}{ll} \frac{-\text{Cos} \, (\lambda \, (\xi - 1)) \, \text{Sin} \, (\lambda \, x)}{\lambda \, \text{Cos} \, \lambda} & 0 \le x < \xi \\ \frac{-\text{Cos} \, (\lambda \, (x - 1)) \, \text{Sin} \, (\lambda \, \xi)}{\lambda \, \text{Cos} \, \lambda} & \xi < x \le 1 \end{array} \tag{71}$$

For λ=0, a simple analogy to this process, or taking the limit of the expression above as λ→0 gives

$$G = \begin{array}{ll} -x & 0 \le x < \xi \\ -\xi & \xi < x \le 1 \end{array} \tag{72}$$

Method 2: Eigenfunction Expansion

Since the eigenfunctions are complete in the space of piecewise continuous functions, we can look for a solution of the form

$$G = \sum_{n=0}^{\infty} a_n \, \text{Sin} \, (\lambda_n \, x) \tag{73}$$

Since the eigenfunctions come from a regular S-L eigenvalue problem we have an orthogonality condition

$$\int_0^1 \text{Sin} \, (\lambda_n \, x) \, \text{Sin} \, (\lambda_m \, x) \, dx = \begin{array}{ll} 0 & n \ne m \\ 1/2 & n = m \end{array} \tag{74}$$

So we have

$$a_n = 2 \int_0^1 \text{Sin} \, (\lambda_n \, x) \, G \, (x) \, dx \tag{75}$$

Multiplying both sides of the ODE by one of the eigenfunctions and integrating gives

$$\int_0^1 \text{Sin} \, (\lambda_n \, x) \, G'' \, dx + \lambda^2 \, \frac{1}{2} \, a_n = \text{Sin} \, (\lambda_n \, \xi) \tag{76}$$

Integrating twice by parts gives

$$\begin{aligned} \int_0^1 \text{Sin} \, (\lambda_n \, x) \, G'' \, dx = & \\ (\text{Sin} \, (\lambda_n \, x) \, G' - \lambda_n \, \text{Cos} \, (\lambda_n \, x) \, G)_{x=1} &- (\text{Sin} \, (\lambda_n \, x) \, G' - \lambda_n \, \text{Cos} \, (\lambda_n \, x) \, G)_{x=0} - \lambda_n^2 \int_0^1 \text{Sin} \, (\lambda_n \, x) \, G \, dx \end{aligned} \tag{77}$$

Simplifying

$$\int_0^1 \text{Sin} \, (\lambda_n \, x) \, G'' \, dx = -\lambda_n^2 \, \frac{1}{2} \, a_n \tag{78}$$

So we have

$$-\lambda_n^2 \, \frac{1}{2} \, a_n + \lambda^2 \, \frac{1}{2} \, a_n = \text{Sin} \, (\lambda_n \, \xi) \tag{79}$$

Which simplifies to

$$a_n = \frac{2}{\lambda^2 - \lambda_n^2} \, \mathrm{Sin}\,(\lambda_n \, \xi) \tag{80}$$

So the Green's function is

$$G = \sum_{n=0}^{\infty} \frac{2}{\lambda^2 - \lambda_n^2} \, \mathrm{Sin}\,(\lambda_n \, \xi)\,\mathrm{Sin}\,(\lambda_n \, x) \tag{81}$$

$$\lambda_n = (2\,n + 1)\,\pi / 2$$

c)

$$y'' + \lambda^2 \, y = f(x)$$
$$y(0) = y'(1) = 0 \tag{82}$$

We suspect that the solution is of the form

$$y = \int_0^1 f(\xi)\,G\,(x \mid \xi)\,d\xi \tag{83}$$

It is easy to check that this is a solution (assuming that derivatives can be passed through the integral)

$$y'' = \int_0^1 f(\xi)\,G_{xx}\,(x \mid \xi)\,d\xi = \int_0^1 f(\xi)\,(\delta\,(x - \xi) - \lambda^2\,G\,(x \mid \xi))\,d\xi = f(x) - \lambda^2\,y \tag{84}$$

$$y(0) = \int_0^1 f(\xi)\,G\,(0 \mid \xi)\,d\xi = \int_0^1 f(\xi)\,0\,d\xi = 0$$
$$y'(1) = \int_0^1 f(\xi)\,G_x\,(1 \mid \xi)\,d\xi = \int_0^1 f(\xi)\,0\,d\xi = 0 \tag{85}$$

Method 1: The Usual Approach

We can write out the expression for y and simplify

for $\lambda \neq 0$

$$y = -\frac{\mathrm{Cos}\,(\lambda\,(x - 1))}{\lambda\,\mathrm{Cos}\,\lambda}\int_0^x f(\xi)\,\mathrm{Sin}\,(\lambda\,\xi)\,d\xi - \frac{\mathrm{Sin}\,(\lambda\,x)}{\lambda\,\mathrm{Cos}\,\lambda}\int_x^1 f(\xi)\,\mathrm{Cos}\,(\lambda\,(\xi - 1))\,d\xi \tag{86}$$

for $\lambda = 0$

$$y = -x\int_x^1 f(\xi)\,d\xi - \int_0^x f(\xi)\,\xi\,d\xi \tag{87}$$

Since λ isn't an eigenvalue, Cos $\lambda \neq 0$, so solutions exist in this form provided that the integrals exist.

Method 2: Eigenfunction Expansion

We can write out the expression for y and simplify (assuming that integration can be interchanged with the sum)

$$y = \int_0^1 f(\xi)\sum_{n=0}^{\infty} \frac{2}{\lambda^2 - \lambda_n^2}\,\mathrm{Sin}\,(\lambda_n\,\xi)\,\mathrm{Sin}\,(\lambda_n\,x)\,d\xi = \sum_{n=0}^{\infty} \frac{f_n}{\lambda^2 - \lambda_n^2}\,\mathrm{Sin}\,(\lambda_n\,x) \tag{88}$$

where f_n are the coefficients

$$f_n = 2\int_0^1 f(x)\,\mathrm{Sin}\,(\lambda_n\,x)\,dx \tag{89}$$

A solution of this form will exist provided that the integrals do.

Using either method, the solution will be unique. Suppose that it weren't unique, i.e. there are (at least) two solutions

$$y'' + \lambda^2\, y = f(x)$$
$$y(0) = y'(1) = 0 \qquad\qquad (90$$
$$z'' + \lambda^2\, z = f(x)$$
$$z(0) = z'(1) = 0$$

Consider the difference of these, w=y-z. As you may check, by definition w is the solution to

$$w'' + \lambda^2\, w = 0$$
$$w(0) = w'(1) = 0 \qquad\qquad (91)$$

For λ not an eigenvalue, this BVP has only the trivial solution w(x)=0. Hence y=z so there can't be more than one solution.

d)

Students may explain this part by simply referring to equation 12 of a handout Dr. Phinney distributed, or they may use one of the following mehtods.

Method 1: The Usual Approach

$$y = -\frac{\text{Cos}\,(\lambda\,(x-1))}{\lambda\,\text{Cos}\,\lambda}\int_0^x f(\xi)\,\text{Sin}\,(\lambda\,\xi)\,d\xi - \frac{\text{Sin}\,(\lambda\,x)}{\lambda\,\text{Cos}\,\lambda}\int_x^1 f(\xi)\,\text{Cos}\,(\lambda\,(\xi-1))\,d\xi \qquad (92)$$

If we expand the Cosine terms and simplify we get

$$y = -\frac{\text{Sin}\,(\lambda\,x)\,\text{Sin}\,(\lambda)}{\lambda\,\text{Cos}\,\lambda}\int_0^1 f(\xi)\,\text{Sin}\,(\lambda\,\xi)\,d\xi -$$
$$\frac{\text{Sin}\,(\lambda\,x)}{\lambda}\int_x^1 f(\xi)\,\text{Cos}\,(\lambda\,\xi)\,d\xi - \frac{\text{Cos}\,(\lambda\,x)}{\lambda}\int_0^x f(\xi)\,\text{Sin}\,(\lambda\,\xi)\,d\xi \qquad (93)$$

These last two terms will be bounded when λ is one of the eigenvalues. If we were to let $\lambda \to \lambda_n$ in the first term the denominator would vanish. So the only way we can get a finite limit as we let $\lambda \to \lambda_n$ is if we require

$$\int_0^1 f(\xi)\,\text{Sin}\,(\lambda_n\,\xi)\,d\xi = 0 \qquad\qquad (94)$$

This solution isn't unique, since

$$z = y + A y_n \qquad\qquad (95)$$

is also a solution for any A as you may check.

Method 2: Eigenfunction Expansion

$$y = \sum_{n=0}^{\infty} \frac{f_n}{\lambda^2 - \lambda_n{}^2}\,\text{Sin}\,(\lambda_n\,x)$$
$$f_n = 2\int_0^1 f(x)\,\text{Sin}\,(\lambda_n\,x)\,dx \qquad\qquad (96)$$

If λ is one of the eigenvalues, say λ_m, then the solution will only be valid if

$$f_m = 2\int_0^1 f(x)\,\text{Sin}\,(\lambda_m\,x)\,dx = 0 \qquad\qquad (97)$$

Notice that by adding any multiple of y_n to y we still have a solution to the PDE and boundary conditions. So the solution isn't unique since any function of the form

$$z = y + A y_n \qquad\qquad (98)$$

for any constant A will give a solution.

Extra stuff not to grade on

We might suspect that our solution method itself is flawed and that maybe the Green's function method fails to produce a solution when $\lambda = \lambda_m$ unless f has certain properties. It turns out that the condition we found is quite general and isn't a result of the green's function method. Define Ly=y'' and suppose y is a solution to the ODE, then since, with the given boundary conditions, L is self-adjoint we have

$$(f, y_m) =$$
$$(Ly + \lambda_m^2 \, y, y_m) = (Ly, y_m) + \lambda_m^2 \, (y, y_m) = (y, Ly_m) + \lambda_m^2 \, (y, y_m) = (\lambda_m^2 - \lambda_m^2) \, (y, y_m) = 0 \qquad (99)$$

So the only way there can be a solution to the ODE is if f is orthogonal to the eigenfunction corresponding to λ_m. In deriving this we didn't use any Green's functions.

e)

Students may explain this part by simply referring to equation 8 of a handout Dr. Phinney dstributed, or they may use one of the following mehtods.

Method 1: The Usual Approach
Recall the expression we found for the solution in part (d)

$$y = -\frac{\mathrm{Sin}\,(\lambda_n \, x)}{\lambda_n} \int_x^1 f(\xi)\,\mathrm{Cos}\,(\lambda_n \, \xi)\,d\xi - \frac{\mathrm{Cos}\,(\lambda_n \, x)}{\lambda_n} \int_0^x f(\xi)\,\mathrm{Sin}\,(\lambda_n \, \xi)\,d\xi \qquad (100)$$

After some algebra

$$y \rightarrow \frac{1}{\lambda_n} \int_x^1 f(\xi)\,\mathrm{Sin}\,(\lambda_n\,(\xi - x))\,d\xi \qquad (101)$$

So the (non-unique) solution is

$$\frac{1}{\lambda_n} \int_x^1 f(\xi)\,\mathrm{Sin}\,(\lambda_n\,(\xi - x))\,d\xi + A\,\mathrm{Sin}\,(\lambda_n\,x) \qquad (102)$$

Method 2: Eigenfunction Expansion
Recall the expression we found for the solution in part (c)

$$y = \sum_{k=0}^{\infty} \frac{f_k}{\lambda^2 - \lambda_k^2} \, \mathrm{Sin}\,(\lambda_k \, x)$$
$$f_k = 2 \int_0^1 f(x)\,\mathrm{Sin}\,(\lambda_k \, x)\,dx \qquad (103)$$

In (d) we made the restriction that f_n=0 and also observed the non-uniqueness. So our (non-unique) solution is

$$\sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{f_k}{\lambda^2 - \lambda_k^2} \, \mathrm{Sin}\,(\lambda_k \, x) + A\,\mathrm{Sin}\,(\lambda_n \, x) \qquad (104)$$

Problem 5 (2×15 points extra credit)

$$\frac{1}{c^2}\,u_{tt} = \frac{1}{r}\,(r\,u_r)_r$$
$$u\,(r, 0) = f\,(r)$$
$$u_t\,(r, 0) = 0 \qquad (105)$$
$$u\,(0, t) = \mathrm{finite}$$
$$u\,(\infty, 0) = 0$$

a)

Given:

$$u(r, t \mid a) = \sum_{m=1}^{\infty} C_m \, J_0 \, (\alpha_m \, r/a) \, \text{Cos} \, (\alpha_m \, ct/a)$$

$$C_m = \frac{\int_0^a r \, f(r) \, J_0 \, (\alpha_m \, r/a) \, dr}{\int_0^a r \, J_0^2 \, (\alpha_m \, r/a) \, dr}$$

$$J_0 \, (\alpha_m) = 0$$

$$\alpha_m \to \pi \, (m - 1/4)$$

$$\int_0^1 x \, J_0^2 \, (\alpha_m \, x) \, dr = J_1^2 \, (\alpha_m)/2 = J_0^{\prime 2} \, (\alpha_m)/2 \to 1/(m \pi^2) = 1/(\alpha_m \, \pi + \pi^2/4)$$

$$C_m = \frac{\int_0^a r \, f(r) \, J_0 \, (\alpha_m \, r/a) \, dr}{\int_0^a r \, J_0^2 \, (\alpha_m \, r/a) \, dr} \tag{107}$$

Define

$$\omega_m = \alpha_m / a \tag{108}$$

Notice

$$\Delta \omega_m = \frac{\alpha_{m+1} - \alpha_m}{a} \to \frac{\pi}{a} \tag{109}$$

Let us write u as

$$u = \sum_{m=1}^{\infty} \phi_m \, J_0 \, (\omega_m \, r) \, \text{Cos} \, (\omega_m \, ct) \, \frac{\pi}{a}$$

$$\phi_m = \frac{a}{\pi} \, C_m = \frac{a}{\pi} \, \frac{\int_0^a r \, f(r) \, J_0 \, (\alpha_m \, r/a) \, dr}{\int_0^a r \, J_0^2 \, (\alpha_m \, r/a) \, dr} = \frac{1}{\pi} \, \frac{1}{a \, J_0^{\prime 2} \, (\alpha_m)/2} \int_0^a r \, f(r) \, J_0 \, (\omega_m \, r) \, dr \tag{110}$$

Now, as a→∞, we have

$$\phi_m \to \phi(\omega) = \omega \int_0^\infty r \, f(r) \, J_0 \, (\omega \, r) \, dr$$

$$u \to \int_0^\infty \phi(\omega) \, J_0 \, (\omega \, r) \, \text{Cos} \, (\omega \, ct) \, d\omega \tag{111}$$

Setting t=0 we have the transform pair described by

$$f(r) = \int_0^\infty \omega \left(\int_0^\infty s \, f(s) \, J_0 \, (\omega \, s) \, ds \right) J_0 \, (\omega \, r) \, d\omega \tag{112}$$

b)

$$\frac{1}{c^2} \, u_{tt} = u_{xx} + u_{yy}$$

$$u(x, y, 0) = f(x, y)$$

$$u_t(x, y, 0) = 0 \tag{113}$$

$$u(0, t) = \text{finite}$$

$$u(\infty, 0) = 0$$

$$\bar{u} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2 \pi i \, (f_x, f_y) \cdot (x, y)} \, u(x, y, t) \, dx \, dy \tag{114}$$

Fourier transform the equation. The transform of the Laplacian is done just as in problem 3

$$\bar{u}_{tt} = -4 \pi^2 \, c^2 \, (f_x^2 + f_y^2) \, \bar{u} \tag{115}$$

So the solution is

$$\bar{u} = A \, \mathrm{Cos}\left(2\,\pi\,c\,\sqrt{f_x^2 + f_y^2}\ t\right) + B \, \mathrm{Sin}\left(2\,\pi\,c\,\sqrt{f_x^2 + f_y^2}\ t\right) \tag{116}$$

where A and B are determined by the initial data

$$\bar{u}(f_x, f_y, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\,\pi\,i\,(f_x, f_y)\cdot(x,y)}\, f(x, y)\, dx\, dy \tag{117}$$
$$\bar{u}_t(f_x, f_y, 0) = 0$$

So we have:

$$\bar{u} = \mathrm{Cos}\left(2\,\pi\,c\,\sqrt{f_x^2 + f_y^2}\ t\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\,\pi\,i\,(f_x, f_y)\cdot(x,y)}\, f(x, y)\, dx\, dy \tag{118}$$

or

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\,\pi\,i\,(f_x, f_y)\cdot(x,y)}\left(\mathrm{Cos}\left(2\,\pi\,c\,\sqrt{f_x^2 + f_y^2}\ t\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\,\pi\,i\,(f_x, f_y)\cdot(u,v)}\, f(u, v)\, du\, dv\right) df_x\, df_y \tag{119}$$

One way to perform these integrals is to change to polar coordinates

$$\begin{aligned} u &= s\,\mathrm{Cos}\,\phi \\ v &= s\,\mathrm{Sin}\,\phi \\ f_x &= z\,\mathrm{Cos}\,\psi \\ f_y &= z\,\mathrm{Sin}\,\psi \\ x &= r\,\mathrm{Cos}\,\theta \\ y &= r\,\mathrm{Sin}\,\theta \end{aligned} \tag{120}$$

$$u = \int_0^{2\pi}\int_0^{\infty} z\, e^{2\,\pi\,i\,r\,z\,\mathrm{Cos}\,(\theta-\psi)}\, \mathrm{Cos}\,(2\,\pi\,c\,z\,t)\, A(\psi, z)\, dz\, d\psi \tag{121}$$

$$A(\psi, z) = \int_0^{2\pi}\int_0^{\infty} s\, e^{-2\,\pi\,i\,s\,z\,\mathrm{Cos}\,(\phi-\psi)}\, f(s)\, ds\, d\phi \tag{122}$$

Consider changing the order of integration in the expression for A

$$A(\psi, z) = \int_0^{\infty} s\, f(s)\left(\int_0^{2\pi} e^{-2\,\pi\,i\,s\,z\,\mathrm{Cos}\,(\phi-\psi)}\, d\phi\right) ds \tag{123}$$

By writing out the real and imaginary parts of the integrand, exploiting symmetry, and using the definition of the order zero Bessel function of the first kind

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \mathrm{Cos}\,(x\,\mathrm{Cos}\,t)\, dt \tag{124}$$

We are able to calculate

$$\begin{aligned} \int_0^{2\pi}& \mathrm{Cos}\,(2\,\pi\,s\,z\,\mathrm{Cos}\,(\phi-\psi))\, d\phi = \\ &2\int_0^{\pi} \mathrm{Cos}\,(2\,\pi\,s\,z\,\mathrm{Cos}\,\phi)\, d\phi = 2\,\pi\left(\frac{1}{\pi}\int_0^{\pi} \mathrm{Cos}\,(2\,\pi\,s\,z\,\mathrm{Cos}\,\phi)\, d\phi\right) = 2\,\pi\,J_0\,(2\,\pi\,s\,z) \\ \int_0^{2\pi}& \mathrm{Sin}\,(2\,\pi\,s\,z\,\mathrm{Cos}\,(\phi-\psi))\, d\phi = 0 \end{aligned} \tag{125}$$

These give a simpler way to write A

$$A(\psi, z) = 2\,\pi\int_0^{\infty} s\, f(s)\, J_0\,(2\,\pi\,s\,z)\, ds \tag{126}$$

Notice that A is actually only a function of z. Inserting this into the integral for u and changing the order of integration gives

$$u = \int_0^\infty z \, \text{Cos}\,(2\,\pi\,c\,z\,t)\,A\,(z) \left(\int_0^{2\,\pi} e^{2\,\pi\,i\,r\,z\,\text{Cos}\,(\theta-\psi)} \, d\psi \right) dz \tag{127}$$

This inner integral was just calculated (except with a + instead of a - in the exponential) so the inner integral is

$$2\,\pi\,J_0\,(-2\,\pi\,r\,z) \tag{128}$$

From the definition, we see that the Bessel function is an even function so our formula for u is

$$u = 4\,\pi^2 \int_0^\infty z \left(\int_0^\infty s\,f\,(s)\,J_0\,(2\,\pi\,s\,z)\,ds \right) \text{Cos}\,(2\,\pi\,c\,z\,t)\,J_0\,(2\,\pi\,r\,z)\,dz \tag{129}$$

Setting t=0 gives the Hankel transform pair

$$f\,(r) = 4\,\pi^2 \int_0^\infty z \left(\int_0^\infty s\,f\,(s)\,J_0\,(2\,\pi\,s\,z)\,ds \right) J_0\,(2\,\pi\,r\,z)\,dz \tag{130}$$

Changing the integration variable

$$\omega = 2\,\pi\,z \tag{131}$$

gives the expression derived in part (a)

$$f\,(r) = \int_0^\infty \omega \left(\int_0^\infty s\,f\,(s)\,J_0\,(\omega\,s)\,ds \right) J_0\,(\omega\,r)\,d\omega \tag{132}$$