Problem Set 6

Feb 20, 2004 — Due Feb 27, 2004
ACM 95b/100b — 3pm at Firestone 303
E. Sterl Phinney — (2 pts) Include grading section number

Useful Readings: For separation of variables, Sections 8.1 and 8.3 of Arfken, Chapter 14 of Carrier and Pearson, or Boyce and DiPrima section 10.5. Bits of Arfken chapter 11 (Bessel functions) and 14 (Fourier series) may be helpful if you get stuck in problem 3. *The same rules apply here regarding computer algebra and plotting as were stated in Problem Set 5.*

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1. (4×5 points) For each of the following partial differential equations, what system of ordinary differential equations results when separation of variables is applied? (Make sure to clearly define your separation constants.)

   a) $u_{tt} = c^2 u_{xx}$ one-dimensional wave equation, c=wave speed

   b) $u_{xx} + u_{yy} + u_{zz} = 0$ Laplace's equation in Cartesian 3-space

   c) $u_t = k u_{xx} - c u_x$ one-dimensional advection-diffusion equation (k=diffusion coefficient, c=advection speed)

   d) $v_t + r x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = rv$ Black-Scholes PDE
   $v(x,t)$=stock option price, x=price of the option's underlying stock, t=time
   Constants: r=interest rate, σ= volatility of the underlying stock

2. (3×5 points) What systems of ordinary differential equations arise when separation of variables is applied to the heat equation $u_t = \kappa \nabla^2 u$ in these geometries? (Make sure to clearly define your separation constants.).

   a) Cartesian 3-space (x, y, z)

   b) Spherical coordinates with spherical symmetry (no θ and ϕ dependence)

   c) Spherical coordinates

   Note: In spherical coordinates (r, θ, ϕ), where θ is the co-latitude and ϕ the longitude,

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

3. (5×5 points) In class we solved for the motion of an axisymmetric circular drumhead of radius a clamped around the rim and starting from rest in some (axisymmetric) state $u(r,0) = f(r)$. We obtained a solution as an infinite sum of products: $u(r,t) = \sum_{k=1}^{\infty} a_k J_0(\alpha_k r/a)\cos(\alpha_k c t/a)$ where $a_k = 2/(J_1^2(a_k)a^2)\int_0^a r f(r) J_0(\alpha_k r/a)dr$, and the α_k are the roots of the Bessel function $J_0(r)$.

   In this problem you will abandon the requirement of axisymmetry, and solve for the motion of the drumhead subjected to an arbitrary initial state (for simplicity we'll start it from rest, as we did in class).

   Thus you are to solve for the $u(r, \varphi, t)$ which satisfies:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad u(r,\varphi,0) = f(r,\varphi), \quad u_t(r,\varphi,0) = 0; \quad u(a,\varphi,t) = 0, \quad u(0,\varphi,t) = \text{finite} \,. \quad (1)$$

In polar coordinates (r, φ),

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2},$$

a) Find the ODEs that arise from separation of variables.

b) Show that the φ equation and its physical boundary conditions form a periodic Sturm-Liouville problem, and find the eigenvalues and eigenfunctions (which you already know form a complete basis, even though it isn't a regular Sturm-Liouville problem covered by the theorem given in class).

c) Taking the φ eigenvalues from the previous part, show that the radial equation leads to the following singular Sturm-Liouville problem for the radial eigenfunctions $R(r)$:

$$\frac{d}{dr}(r\frac{dR}{dr}) - \frac{n^2}{r}R + \lambda r R = 0, \quad R(a) = 0, \quad R(0) = \text{finite} \tag{2}$$

where n is an integer (an eigenvalue of part (b)). Show that this eigenvalue problem, known as Bessel's equation, is of Sturm-Liouville form, and identify $p(r)$, the weight $w(r)$, and write down the orthogonality condition you expect to be satisfied by the eigenfunctions.

d) What is the indicial equation for the Bessel equation of part (c)? Show that it is such that you expect the general solution to the differential equation to have a solution of Frobenius form (finite or vanishing at $r = 0$), known as $J_n(\sqrt{\lambda}r)$, and the other (known as $Y_n(\sqrt{\lambda}r)$) to diverge as $r \to 0$. Argue that the latter cannot contribute here. Thus show that the relevant eigenvalues are $\lambda_{nj} = (\alpha_{nj}/a)^2$, where α_{nj} is the jth root of $J_n(x) = 0$. For the next part you may assume (as can be proven) that for *any* n, the eigenfunctions $y_j(r) \equiv J_n(\alpha_{nj}r/a)$ form a complete basis.

e) Solve the remaining differential equation for the time dependence, and thus show that any solution $u(r, \varphi, t)$ can be expanded as

$$u(r, \varphi, t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} A_{nj} J_n(\alpha_{nj} r/a) \left[\sin n\varphi + B_n \cos n\varphi\right] \cos(\alpha_{nj} ct/a) \ . \tag{3}$$

Give an expression for the A_{nj} and B_n in terms of integrals of known functions (including the initial condition function $f(r, \varphi)$, J_n's and trig functions). You may find it helpful (and more honest in case some of the A_{nj} are zero) to define $A_{nj}B_n = C_{nj}$ and solve for the C_{nj}. The solution was written in this way just to emphasize that the ratio of coefficients of sine and cosine does not depend on j. Notice that for $n \neq 0$ there are always two independent modes (sine and cosine) with the same frequency. Beware: the term with $n = 0$ has a slightly different normalization condition than the others, so check it separately. Hint: in determining the coefficients, it is much easier to use orthogonality in one order of the double integration in φ and r than in the other.

4. (5×5 points) We now return to PS5 Problem 4: a flexible string of length L, uniform mass per unit length μ and constant tension T subject to an external force per unit length $f(x, t)$, whose small vibrations obey

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} + f(x, t), \quad y(0) = y(L) = 0 \ . \tag{4}$$

Here consider the time-dependent problem in which $f(x,t) = 0$ for $t > 0$ (i.e. the force is used to start the string moving, but is removed after that).

We consider two types of initial conditions:

P: *pluck* String is let go from rest at some equilibrium shape $Y(x)$: $y(x,0) = Y(x)$ and $\partial y/\partial t \equiv y_t(x,0) = 0$. We will here consider $Y(x)$ to be the steady-state equilibrium solution $y(x)$ you found in PS5, part 4a (force applied by a delta-function pick at a specified place along the string). This is an appropriate model for harpsichord and guitar strings.

H: *light hammer* String is set in motion by a strike from a hammer which quickly slows to a stop after it imparts its kinetic energy to a small length of string. In the limit of a very light hammer, this corresponds to $y(x,0) = 0$ and $y_t(x,0) = V(x)$. We will here consider $V(x) = P_h/\mu\delta(x-\xi)$, where P_h is the hammer's initial momentum, and ξ its strike position. This is an appropriate model for the lowest octave of a piano. [Interesting fact not necessary for this problem: this approximation becomes bad above C2 because for these higher frequencies, the piano hammer stays in contact with the string for longer than the time for the wave to return from the near post -and in the upper registers, even for longer than a wave period].

You separated variables for this in Problem 1a. In PS5 Problem 4c, you showed that the resulting spatial differential equation was of Sturm-Liouville form.

a) Use the step-by-step procedure outlined in class to find $y(x,t)$. i.e. first expand $y(x,t)$ at fixed t in terms of the eigenfunctions of the spatial Sturm-Liouville problem times coefficients $a_n(t)$ (which will depend on t). Then use the PDE to determine the time-dependence of $a_n(t)$, and the initial conditions to determine $a_n(0)$. You should give explicit solutions for initial conditions of type P and of type H. (i.e. solutions of the form $y = \sum A_n X_n(x) T_n(t)$, with A_n determined by explicit integrals over $Y(x)$ and $V(x)$ respectively).

b) Write out explicitly the (sum of eigenfunctions) solution for the initial condition of type P with $Y(x)$ given in PS5, Problem 4d (i.e. $L = 1$, string plucked at $x = (1/7)L$). The human ear carries frequencies separated by more than a factor of 1.18 (the 'critical bandwidth') on separate neural paths, and those separated by less on the same ones. This gives rise to the sensation of dissonance in tones with frequency ratios less than a minor third (6/5). Given this, what about the amplitudes of different frequencies in your solution suggests that $x = (1/7)L$ would be an especially good choice for plucking position? This is often cited in elementary texts as the actual position of piano hammers.

c) Pluck a rubber band, and sketch how the vibrating band looks to your eye. Now set $T/\mu = c^2 = 1$ and plot the sum of the first 50 terms of your solution from part (b) for $t = 0$ to $t = 1$ in steps of 0.05. Surprised? What is the period (in time) of the waveform? [Note: you may not wish to hand in all 20 separate plots, but instead select a few important ones and show what happens in beteween with arrows indicating direction of motion, or an overlay (possibly with vertical offsets) of the plots, or a space-time diagram. Use your imagination and good sense.]

d) Now write out the sums of eigenfunction solution for initial condition H, setting $L = 1$, $P_h/\mu = 1$ and $T/\mu = c^2 = 1$, and $\xi = (1/7)L$. Is the latter an optimal (from the point of view of hearing least self-dissonance) position for hammering as it is for plucking?

e) Plot the sum of the first 50 terms of your solution from part (b) for $t = 0$ to $t = 2$ in steps of about 0.05 (it is suggested to do several of these separately, rather than overplotting them, so you can see what is going on. Or you may prefer to make an animated movie). Surprised? What is the period (in time) of the waveform? For what to hand in, see note to part (c).

f) (1 point optional extra credit for the interested): on your instructor's piano, the C4 [middle C, 261.6Hz] string is 26 inches long, and the hammer strikes 3 inches from the point where the string is anchored. This is *not* 26/7=3.7 inches. However the hammer is in contact with the string for about 2ms. Compare this to the time it takes the wave you found in (e) to return to the hammer so see why real piano string-hammer interactions are more complicated than we assumed, and the actual location that best suppresses the undesirable 7th harmonic is not really 1/7.]

5. (3×4 points) In class it was stated that while it was often OK to substitute eigenfunction expansions into PDEs and equate term by term, it was not *always* ok, and the only foolproof method was to multiply the PDE by the eigenfunction and integrate by parts. Here you will find out how things can go wrong with the simple approach.

   Consider solving Laplace's equation in two dimensions in the unit box:

$$u_{xx} + u_{yy} = 0, \quad u(x,1) = u(0,y) = u(1,y) = 0, \quad u(x,0) = f(x) \tag{5}$$

   [This gives for example the electric potential u inside a box three sides of which are grounded, while the fourth side has an applied potential $f(x)$. This could also be posed as giving the equilibrium temperature of a square pan kept heated on one side. Physical intutition tells you that the solution is NOT $u(x,y) = 0$]. The set of functions $\sin(n\pi y)$ solve a regular Sturm-Liouville problem, so form a complete basis for square-integrable functions on [0,1]. Thus we choose to represent $u(x,y)$ as

$$u(x,y) = \sum_{n=1}^{\infty} c_n(x) \sin(n\pi y) \ . \tag{6}$$

   a) Show that the boundary conditions give $c_n(0) = c_n(1) = 0$.

   b) Substitute eq (6) into the PDE, differentiate term by term, and use orthogonality to show that $d^2 c_n(x)/dx^2 - n^2\pi^2 c_n = 0$.

   c) Use the boundary conditions of part (a) to show that the solution to the equations of part (b) are $c_n(x) = 0$, and hence $u(x,y) = 0$, which is *wrong* (as obvious from the physical system)!!!

   d) (optional 10 points extra credit) Since y'all don't like longer calculations, I'll just state that the correct solution is

$$c_n(x) = \frac{2}{\sinh n\pi} \left[\int_0^x f(t) \sinh n\pi t \sinh n\pi(1-x)\, dt + \int_x^1 f(t) \sinh n\pi x \sinh n\pi(1-t)\, dt \right] \tag{7}$$

   If you are interested, you can derive this by the (always) correct procedure of multiplying the PDE by an eigenfunction $\sin n\pi y$ and integrating dy by parts twice, using the BC's, and solving the resulting second order inhomogeneous linear ODE for $c_n(x)$. Can you see what went wrong in the simple-minded approach?

Total points: 99