Problem Set 2 (re-revised 1/19/2004 18:00h)

January 16, 2004
ACM 95b/100b
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Due January 23, 2004
3pm in Firestone 303
(2 pts) Include grading section number

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1. (3×5 points) Solve the following linear initial value problems. Sketch the solution and describe the behaviour as the independent variable $x \to \infty$.

   a) $y'' + 6y' + 5y = 0$ with $y(0) = 3,\ y'(0) = -2$

   b) $y'' + 2y' + 5y = 0$ with $y(0) = 0,\ y'(0) = 5$

   c) $y'' - 2y' + y = 0$ with $y(0) = 1,\ y'(0) = 0$

2. (2×7 points) Consider the Legendre equation of order one (the Legendre equation arises when separating Laplace's equation in spherical coordinates —e.g. in electromagnetism, quantum mechanics, and potential theory of nonspherical bodies —e.g. planets),

$$(1 - x^2)y''(x) - 2xy' + 2y = 0 \quad -1 < x < 1 , \tag{1}$$

   a) find (up to a constant multiplicative factor) the Wronskian of two solutions without solving the ODE.

   b) verify that $y(x) = x$ is a solution and use reduction of order to find a general solution to the ODE.

3. (10 points) [Variation of parameters.] Find a particular solution to

$$t^2 y'' - 2y = 3t^2 - 1, \quad t > 0 \tag{2}$$

Hint: notice that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are linearly independent solutions to the homogeneous problem.

4. (10 points) [Variation of parameters] Show that the solution of the IVP

$$y'' + y = g(x), \quad \text{with} \quad y(0) = 0, \quad y'(0) = 0, \tag{3}$$

has the form

$$y = \int_0^x \sin(x - s)g(s)ds . \tag{4}$$

5. (4×7 points) Use the method of undetermined coefficients (see Note below) to find the general solution of the following ODE describing a driven harmonic oscillator:

$$L(y) \equiv y'' + y = x^2 + e^{-ax}\sin(\omega x) \tag{5}$$

   a) for real ω and real $a \neq 0$.

   b) for $a = 0$ and $\omega = 1$. What is special about this case?

   c) Set $\omega = 1$ and let $a \to 0$ in your answer to (a). Show that this reproduces your answer to (b).

   d) Can you also reproduce your answer to (b) by taking the limit in the other order (i.e. set $a = 0$ and let $\omega \to 1$)?

Note for parts (a,b): the method of undetermined coefficients consists of noticing that the linear differential operator L applied to polynomials, polynomials times exponentials and polynomials times trig functions, returns functions in those same classes. So for a sufficiently general choice of 'guessed solution' consisting of sums of such terms, the coefficients in those sums can be chosen so the sum satisfies the ODE.

6. [2 × 5 points] Identify all the singular points of the following differential equations, and classify them [as regular singular points or essential singularities]. Be sure to consider the point at infinity (i.e investigate the behaviour of the equations with the substitution $z = 1/x$ as $z = 1/x \to 0$).

  a) $(1 - x^2)y'' - xy' + \alpha^2 y = 0$ (Chebyshev)

  b) $x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$ (Bessel)

7. [4 × 7 points] Method of Frobenius

  a) Find the two linearly independent series solutions about $x = 0$ to the Chebyshev differential equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0 \tag{6}$$

  You should check the linear independence by checking that the Wronskian of (the first two terms of) your two series is nonzero for small x.

  b) What is the radius of convergence of your series in (a)? Could you have anticipated this by considering the form of the equation and the theorems given in class?

  c) Substitute $z = 1/x$ in the Chebyshev differential equation, and find the series solution for the resulting equation about $z = 0$ (i.e. about $x = \infty$). There are several special cases, depending on values of the parameter α. To avoid doing too much tedious algebra, you may restrict your answer to consideration of three cases: $\alpha = 0$, $\alpha = 2$ and $\alpha \neq k/2$, where k is an integer. For 7 points extra credit, you may consider the remaining cases (α equal to other integers and half-integers).

  d) What is the radius of convergence of your series in (c)? Could you have anticipated this by considering the form of the equation and the theorems given in class?

Total points: 117