ACM 95b/100b Midterm Exam

February 6, 2004 | Due February 10, 2004
ACM 95b/100b | 3pm in Firestone 307
E. Sterl Phinney | (2 pts) Include grading section number

The honor code is in effect. Please follow all of the following instructions regarding this exam. If you feel unclear about any of these instructions, you are required by the honor code to ask for clarification.

- You may not communicate with any other person (except Prof. Phinney) about the contents of this exam, until both you and the other person (if taking ACM95b/100b) have submitted your exams. The Deans have given a few students extensions on the exam, so please do not talk about it loudly in public, even after the due date.
- The exam must be completed in a single sitting of **4 hours**.
- There are 4 problems, each with four parts. Each part is worth 6 points, except for the first part of the first problem, which is worth 8. The total exam is worth $(8 + 3 \times 6) + 3 \times 4 \times 6 + 2 = 100$ points (the 2 points are for writing your grading section number on the cover of your blue book).
- Calculators, computers and related devices are **not permitted**.
- **Closed book, computer off exam:** with the exception of official lecture handouts and problem set questions from this course, only material written in your own hand may be used during the exam.
- Please write your exam in standard **blue books** and make sure your name and **grading section number** (as assigned by the ACM95b/100b Underground) is clearly written on the front of each blue book.
- You must sign in your completed exam in person to Maria Katsas in Firestone 307 by **3pm on Tuesday February 10**. *Note: Ms. Katsas will not be in her office 2-5pm Monday Feb 9. Please plan your visit accordingly.*

Good luck!

Dialog from a Calvin and Hobbes cartoon dated 3/6/91:

Calvin: You know, I don't think math is a science, I think it's a religion.

Hobbes: A religion?

Calvin: Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one NEW number! No one can say how it happens. You either believe it or you don't. [Pointing at his math book] This whole book is full of things that have to be accepted on faith! It's a religion!

Hobbes: And in the public schools no less. Call a lawyer.

Calvin: [Looking at his homework] As a math atheist, I should be excused from this.

Sorry, you still have to take the midterm.

Please be sure to write your (Underground-assigned) grading section number on the cover of your bluebook, for 2 free points.

The midterm starts on the next page.

1. $(8 + 3 \times 6$ points)

a) Find and classify the singularities of the following differential equations for $y(x)$. Consider only finite (but complex) x —i.e. you need not study the point at infinity.

$$x^3 y'' + x^4 y' + 2y = 0 \tag{1}$$

$$(x^2 - 3x + 2)y'' - xy' + x^2 y = 0 \tag{2}$$

$$y'' + 2y' + 2y = 0 \tag{3}$$

$$(e^x - 1)y'' + xy = 0 \tag{4}$$

b) i. For the equation (2) in part (a), if a power series solution is expanded about $x_0 = 0$, for what values of x can it be guaranteed to converge? State your reasoning.

ii. Answer the same question as in (i), but for equation (4).

c) For the equation (2) of part (a), it is desired to find a series or Frobenius-type expansion of the general solution about the point $x_0 = 1$.

i. Write down and solve the indicial equation for such an expansion.

ii. Can the general solution for $y(x)$ in some small but finite interval about $x_0 = 1$ be written as a pure power series expanded about $x_0 = 1$?

iii. If so, why? If not, what additional functional forms should be added?

d) Solve equation (3) on the interval $x > 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$ by first finding the Laplace transform $Y(s)$ of $y(x)$, and then inverting it. To find the inverse transform you may use shifting theorems and your table of transforms, the Mellin inversion, or a combination of those methods.

2. (4×6 points) Consider the differential equation

$$y'' + (2 - 4x^2)y = 0 \; . \tag{5}$$

It is desired to find two solutions $y_1(x)$ and $y_2(x)$ on the interval $I : 0 \leq x < \infty$ which respectively satisfy the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$, $y_2'(0) = 1$.

a) Find the Wronskian $w(x)$ of $y_1(x)$ and $y_2(x)$ over the interval I.

b) Are $y_1(x)$ and $y_2(x)$ linearly independent on the interval I? Why or why not? In terms of y_1, y_2 and any other functions needed, what is the general solution of the differential equation (5)?

c) Seek the solutions $y_1(x)$ and $y_2(x)$ in the form of power series about $x = 0$, and determine the coefficients of the terms of the power series. You need not do this in full generality: for the purposes of this exam full credit will be given for the first three nonzero terms in each series *and* the recursion relation(s) that determine(s) the subsequent terms in the series.

d) It can be shown (you need not do this for the exam) that the series you found for $y_1(x)$ in part (c) is in fact the expansion of $y_1(x) = \exp(-x^2)$. Use the method of reduction of order (or variation of parameters) to find another representation of $y_2(x)$. You may leave your answer in terms of integrals of elementary functions.

3. (4×6 points)

a) Solve

$$\frac{dy}{dx} - \frac{2}{x}y = x^5, \quad y(1) = 0 \tag{6}$$

for $y(x)$ on the interval $x \geq 1$ by finding an integrating factor.

b) Solve for the Green's function $y_G(x|\xi)$ of the above equation, i.e. for $x \geq 1$ and real $\xi \geq 1$, $y_G(x|\xi)$ satisfies

$$\frac{dy_G}{dx} - \frac{2}{x}y_G = \delta(x - \xi), \quad y_G(1) = 0\ . \tag{7}$$

The δ above is the Dirac delta 'function'. Remember that y_G is a function of x, and the notation $y_G(x|\xi)$ is just to remind you that y_G has an important parameter ξ.

c) Give an expression involving $y_G(x|\xi)$ for the solution to

$$\frac{dy}{dx} - \frac{2}{x}y = f(x), \quad y(1) = 0\ , \tag{8}$$

and verify that for $f(x) = x^5$, this reproduces your answer to part (a). [If you were unable to do part (b), you may get full credit for this part by instead solving equation (8) by variation of parameters. You should still check your result by verifying that the general expression reproduces your answer to part (a) for $f(x) = x^5$].

d) If instead of the initial condition used in part (a) ($y(1) = 0$) we instead asked for $y(0) = 0$ what would be the solution for $x \geq 0$ to

$$\frac{dy}{dx} - \frac{2}{x}y = x^5, \quad y(0) = 0\ ? \tag{9}$$

Is the solution unique?

4. (4×6 points) In the parts below, you may find it useful to recall that functions to be Laplace transformed (or found by inverse tranforms) are defined only for $t \geq 0$, so don't waste your time thinking about $t < 0$.

   a) Let k be a positive real number. Show that the Laplace transform of the 'back-to-back bed of nails function'

   $$f(t) = \sum_{n=0}^{\infty} (-1)^n \delta(t - (2n+1)k) \tag{10}$$

   is $F(s) = 1/(2\cosh(ks))$. The δ above is the Dirac delta 'function'. In case you've forgotten, $\cosh(x) = 1/2(\exp(x)+\exp(-x))$ and $\sinh(x) = 1/2(\exp(x)-\exp(-x))$.

   b) Use properties of the Laplace transform to relate $g(t) = \mathcal{L}^{-1}(G(s))$, where $G(s) = F(s)/s$ to some integral of $f(t) = \mathcal{L}^{-1}(F(s))$, where $f(t)$ is the function of part (a). Sketch $g(t)$, and be sure to label both your axes with numerical values.

   c) Use the Mellin inversion formula to compute the inverse Laplace transform of $G(s) = 1/(2s\cosh(ks))$. [Hint: one step is to show that all the roots of $\cosh(ks) = 0$ are pure imaginary —i.e. have zero real part]. Evaluate any residues that arise. You should find that your answer can be written as an infinite sum of real-valued trigonometric functions.

   d) The sum you found in part (c) and the function you found and sketched in part (b) are just two different ways of inverting the same transform $G(s)$.

      i. Plot the first two terms of the sum you found in (c).

      ii. Do the two functions have the same period?

      iii. {Do not spend more than 5 minutes on this part; rigor is not required, but insight and cleverness will be rewarded:} Do you think the sum of (c) adds up to the $g(t)$ of (b)? Give a graphical, analytic or physical argument to motivate your answer.