ACM95b/100b Lecture Notes

Caltech 2004

The Method of Frobenius

Consider the equation

$$x^2y'' + xp(x)y' + q(x)y = 0, \tag{1}$$

where $x = 0$ is a regular singular point. Then $p(x)$ and $q(x)$ are analytic at the origin and have convergent power series expansions

$$p(x) = \sum_{k=0}^{\infty} p_k x^k, \qquad q(x) = \sum_{k=0}^{\infty} q_k x^k, \quad |x| < \rho \tag{2}$$

for some $\rho > 0$. Let r_1, r_2 ($\mathbb{R}(r_1) \geq \mathbb{R}(r_2)$) be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0. \tag{3}$$

Depending on the nature of the roots, there are three forms for the two linearly independent solutions on the intervals $0 < |x| < \rho$. The power series that appear in these solutions are convergent at least in the interval $|x| < \rho$. (Proof: Coddington)

Case 1: Distinct roots not differing by an integer ($r_1 - r_2 \neq N, N \in \mathbb{Z}$)

The two solutions have the form

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k(r_1) x^k \tag{4}$$

$$y_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k \tag{5}$$

where $a_k(r_1)$ and $b_k(r_2)$ are determined by substitution of (4) or (5) into equation (1) to obtain the corresponding recurrence relation.

Case 2: Repeated root ($r_1 = r_2$)

The first solution $y_1(x)$ has form (4) and the second solution has the form

$$y_2(x) = y_1(x) \log x \; + \; x^{r_1} \sum_{k=1}^{\infty} b_k(r_1) x^k. \tag{6}$$

Note that the term $k = 0$ is ommitted as it would just give a multiple of $y_1(x)$.

Case 3: Roots differing by an integer ($r_1 - r_2 = N, N \in \mathbb{Z}^+$)

The first solution $y_1(x)$ has form (4) and the second solution has the form

$$y_2(x) = c\, y_1(x) \log x \; + \; x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k. \tag{7}$$

where c may turn out to be zero. The constant $b_N(r_2)$ is arbitrary and may be set to zero. This is evident by writing

$$x^{r_2} \sum_{k=0}^{\infty} b_k(r_2) x^k = b_0 x^{r_2} + ... + b_{N-1} x^{r_2+N-1} + \underbrace{x^{r_1}(b_N + b_{N+1}x + b_{N+2}x^2 + ...)}_{\text{form of } y_1(x)} \tag{8}$$

so we see that $b_N(r_2)$ plays the same role as $a_0(r_1)$ and merely adds multiples of $y_1(x)$ to $y_2(x)$.

Example: Case 1

Consider

$$4xy'' + 2y' + y = 0 \tag{9}$$

so $x = 0$ is a regular singular point with $p(x) = \frac{1}{2}$ and $q(x) = \frac{x}{4}$. The power series in y_1 and y_2 will converge for $|x| < \infty$ since p and q have convergent power series in this interval. By (3), the indicial equation is

$$r(r-1) + \frac{1}{2}r = 0 \quad \Rightarrow \quad r^2 - \frac{1}{2}r = 0 \tag{10}$$

so $r_1 = \frac{1}{2}$ and $r_2 = 0$ (Note: $p_0 = \frac{1}{2}$, $q_0 = 0$). Substituting $y = x^r \sum_{k=0}^{\infty} a_k x^k$ into (9) and shifting the indices of the first two series so all terms are of form x^{k+r} we get

$$4\sum_{k=-1}^{\infty}(k+r+1)(k+r)a_{k+1}x^{k+r} + 2\sum_{k=-1}^{\infty}(k+r+1)a_{k+1}x^{k+r} + \sum_{k=0}^{\infty}a_k x^{k+r} = 0. \tag{11}$$

All coefficients of powers x^{k+r} must equate to zero to obtain a solution. The lowest power is x^{r-1} for $k = -1$ and this yields

$$4r(r-1) + 2r = 0 \quad \Rightarrow \quad r^2 - \frac{1}{2}r = 0 \tag{12}$$

which is just the indicial equation as expected. For $k \geq 0$, we obtain

$$4(k+r+1)(k+r)a_{k+1} + 2(k+r+1)a_{k+1} + a_k = 0 \tag{13}$$

corresponding to the recurrence relation

$$a_{k+1} = \frac{-a_k}{(2k+2r+2)(2k+2r+1)}, \quad k = 0, 1, 2... \tag{14}$$

First Solution: To find y_1 apply (14) with $r = r_1 = \frac{1}{2}$ to get the recurrence relation

$$a_{k+1} = \frac{-a_k}{(2k+3)(2k+2)}, \quad k = 0, 1, 2... \tag{15}$$

Then

$$a_1 = \frac{-a_0}{3 \cdot 2}, \quad a_2 = \frac{-a_1}{5 \cdot 4}, \quad a_3 = \frac{-a_2}{7 \cdot 6}, \quad ... \tag{16}$$

so

$$a_1 = -\frac{a_0}{3!}, \quad a_2 = \frac{a_0}{5!}, \quad a_3 = -\frac{a_0}{7!}, \quad ... \tag{17}$$

Since a_0 is arbitrary, let $a_0 = 1$ so

$$a_k(r_1) = \frac{(-1)^k}{(2k+1)!}, \quad k = 0, 1, 2... \tag{18}$$

and

$$y_1(x) = x^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^k. \tag{19}$$

Second Solution: To find y_2, just apply (14) with $r = r_2 = 0$ to get the recurrence relation

$$b_{k+1} = \frac{-b_k}{(2k+2)(2k+1)}. \tag{20}$$

Letting the arbitrary constant $b_0 = 1$, then

$$b_k(r_2) = \frac{(-1)^k}{(2k)!} \tag{21}$$

so

$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^k. \tag{22}$$

Example: Case 2

Consider

$$Ly \equiv xy'' + y' - y = 0 \tag{23}$$

with $p(x) = 1$ and $q(x) = -x$ and a regular singular point at $x = 0$. The power series in y_1 and y_2 will converge for $|x| < \infty$ since p and q have convergent power series in this interval. The indicial equation is given by

$$r(r-1) + r = 0 \quad \Rightarrow \quad r^2 = 0 \tag{24}$$

so $r_1 = r_2 = 0$.

First solution: Substituting $y = \sum_{k=0}^{\infty} a_k x^k$ into (23) results in

$$\sum_{k=0}^{\infty}(k+1)ka_{k+1}x^k + \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - \sum_{k=0}^{\infty} a_k x^k = 0 \tag{25}$$

after shifting indices in the first two series to express all terms as multiples of x_k. Regrouping terms gives

$$\sum_{k=0}^{\infty}[(k+1)ka_{k+1} + (k+1)a_{k+1} - a_k]x^k = 0, \tag{26}$$

so equating all coefficients of powers of x to zero gives

$$a_{k+1} = \frac{a_k}{(k+1)^2}, \quad k \geq 0. \tag{27}$$

Then for $k \geq 1$

$$a_k = \frac{a_{k-1}}{k^2} = \frac{a_{k-2}}{k^2(k-1)^2} = ... = \frac{a_0}{(k!)^2}. \tag{28}$$

Setting the arbitrary constant $a_0(r_1) = 1$, the first solution is

$$y_1(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k = 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + ... \tag{29}$$

Second solution: Consider substituting

$$y = y_1(x)\log x + \sum_{k=1}^{\infty} b_k x^k \tag{30}$$

into (23). Then

$$y' = \frac{y_1}{x} + y_1' \log x + \frac{d}{dx}\sum_{k=1}^{\infty} b_k x^k \tag{31}$$

and

$$xy'' = x\left[-\frac{y_1}{x^2} + 2\frac{y_1'}{x} + y_1'' \log x + \frac{d^2}{dx^2}\sum_{k=1}^{\infty} b_k x^k\right] \tag{32}$$

so making cancellations we obtain

$$L[y_1]\log x + 2y_1' + L\left[\sum_{k=1}^{\infty} b_k x^k\right] = 0. \tag{33}$$

Now we know $L[y_1]=0$ so this gives

$$L\left[\sum_{k=1}^{\infty} b_k x^k\right] = -2y_1' \tag{34}$$

or in detail after appropriate index shifts to the first and second series

$$b_1 + \sum_{k=1}^{\infty}\left[(k+1)kb_{k+1} + (k+1)b_{k+1} - b_k\right]x^k = -2 - x - \frac{x^2}{6} - ... \tag{35}$$

Equating coefficients gives

$$\begin{aligned} b_1 &= -2 \\ 4b_2 - b_1 &= -1 \\ 9b_3 - b_2 &= -\frac{1}{6} \\ &\vdots \end{aligned}$$

so

$$b_1 = -2, \quad b_2 = -\frac{3}{4}, \quad b_3 = -\frac{11}{108}, \quad ... \tag{36}$$

The second linearly independent solution is then

$$y_2(x) = y_1(x)\log x + \left[-2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - ...\right] \tag{37}$$

Example: Case 3 (log term required)

Consider

$$Ly \equiv xy'' + y = 0 \tag{38}$$

with $p(x)=0$ and $q(x)=x$ and a regular singular point at $x=0$. The power series in y_1 and y_2 will converge for $|x|<\infty$ since p and q have convergent power series in this interval. The indicial equation is given by

$$r(r-1) = 0 \tag{39}$$

so $r_1 = 1$ and $r_2 = 0$.
First solution: Substituting $y = x^r\sum_{k=0}^{\infty} a_k x^k$ into (38) results in

$$\sum_{k=0}^{\infty}(r+k)(r+k-1)a_k x^{r+k-1} + \sum_{k=0}^{\infty} a_k x^{r+k} = 0. \tag{40}$$

Shifting indices in the second series and regrouping terms gives

$$r(r-1)a_0x^{r-1} + \sum_{k=1}^{\infty}[(r+k)(r+k-1)a_k + a_{k-1}]x^{r+k-1} = 0. \tag{41}$$

Setting the coefficient of x^{r-1} to zero we recover the indicial equation with $r_1=1$ and $r_2=0$. Setting all the other coefficients to zero gives the recurrence relation

$$a_k = \frac{-a_{k-1}}{(r+k)(r+k-1)}, \quad k \geq 1. \tag{42}$$

With $r=r_1$ this gives

$$a_k = \frac{-a_{k-1}}{(k+1)k} = \frac{a_{k-2}}{(k+1)k^2(k-1)} = ... = \frac{(-1)^k a_0}{(k+1)(k!)^2}. \tag{43}$$

Setting the arbitrary constant $a_0(r_1) = 1$, the first solution is then

$$y_1(x) = x\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k!)^2} x^k = x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} \pm ... \tag{44}$$

Second solution: First, let's see how we run into trouble if we fail to include the log term in the second solution. The recurrence relation (42) with $r = r_2 = 0$ becomes (with b_k replacing a_k since we are now using $r = r_2$)

$$b_k = \frac{-b_{k-1}}{k(k-1)}, \quad k \geq 1. \tag{45}$$

This formula fails for $k = 1$. As was anticipated, for roots of the form $r_2 - r_1 = N$ with $N \in \mathbb{Z}^+$ it may not be possible to determine b_N if the log term is ommitted from y_2 (in our case $N = 1$). For the second solution consider substituting

$$y = c\, y_1(x) \log x + x^0 \sum_{k=0}^{\infty} b_k x^k \tag{46}$$

into (38) so

$$xy'' = \left[-\frac{cy_1}{x} + 2cy_1' + cxy_1'' \log x + x \frac{d^2}{dx^2} \sum_{k=0}^{\infty} b_k x^k \right]. \tag{47}$$

We then obtain

$$cL[y_1] \log x + 2cy_1' - \frac{cy_1}{x} + L\left[\sum_{k=0}^{\infty} b_k x^k \right] = 0. \tag{48}$$

Now we know $L[y_1] = 0$ so this gives

$$L\left[\sum_{k=0}^{\infty} b_k x^k \right] = -2cy_1' + \frac{cy_1}{x}. \tag{49}$$

Expanding the left hand side gives

$$b_0 + (2b_2 + b_1)x + (6b_3 + b_2)x^2 + (12b_4 + b_3)x^3 + (20b_5 + b_4)x^4 + ... \tag{50}$$

and expanding the right hand side gives

$$-c + \frac{3}{2}cx - \frac{5}{12}cx^2 + \frac{7}{144}cx^3 - \frac{1}{320}cx^4 \pm ... \tag{51}$$

Equating coefficients gives the system of equations

$$\begin{aligned}
b_0 &= -c \\
2b_2 + b_1 &= \frac{3}{2}c \\
6b_3 + b_2 &= -\frac{5}{12}c \\
12b_4 + b_3 &= \frac{7}{144}c \\
&\vdots
\end{aligned}$$

Now $b_0(r_2)$ is an arbitrary constant and $c = -b_0$. Notice that b_1 can also be chosen arbitrarily. This is because it is the coefficient of $x^{r_1} = x^1 = x$ in the series

$$x^{r_2} \sum_{k=0}^{\infty} b_k x^k = b_0 + \underbrace{x^{r_1}(b_1 + b_2 x + b_3 x^2 + ...)}_{\text{form of } y_1(x)}. \tag{52}$$

Consequently, modifying $b_1(r_2)$ changes subsequent coefficients $b_k(r_2)$ for $k > 1$ so as to effectively add a multiple of $y_1(x)$ to $y_2(x)$. In effect, changing b_1 just affects the choice of the arbitrary constant $a_0(r_1)$ that we already chose to be $a_0 = 1$. For convenience, we now choose $b_0 = 1$ and $b_1 = 0$. Then $b_2 = -3/4$, $b_3 = 7/36$, $b_4 = -35/1728$, ... so the second solution is

$$y_2(x) = -y_1(x)\log x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 \pm ...\right]. \tag{53}$$

Example: Case 3 (log term drops out)

Consider

$$Ly \equiv x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{54}$$

with $p(x) = 1$ and $q(x) = (x^2 - \frac{1}{4})$ and a regular singular point at $x = 0$. The power series in y_1 and y_2 will converge for $|x| < \infty$ since p and q have convergent power series in this interval. The indicial equation is given by

$$r(r-1) + r - \frac{1}{4} = 0 \tag{55}$$

so $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$.

First solution: Substituting $y = x^r \sum_{k=0}^{\infty} a_k x^k$ into (54) results in

$$\sum_{k=0}^{\infty}\left[(r+k)(r+k-1) + (r+k) - \frac{1}{4}\right]a_k x^{r+k} + \sum_{k=0}^{\infty} a_k x^{r+k+2} = 0. \tag{56}$$

or shifting indices in the last series

$$\left(r^2 - \frac{1}{4}\right)a_0x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1x^{r+1} + \sum_{k=2}^{\infty}\left\{\left[(r+k^2) - \frac{1}{4}\right]a_k + a_{k-2}]\right\}x^{r+k} = 0. \tag{57}$$

Setting the coefficient of x^r to zero we recover the indicial equation. Setting the other coefficients to zero we find

$$\left[(r+1)^2 - \frac{1}{4}\right]a_1 = 0 \tag{58}$$

and the recurrence relation

$$\left[(r+k)^2 - \frac{1}{4}\right]a_k = -a_{k-2}, \quad k \geq 2. \tag{59}$$

For $r_1 = \frac{1}{2}$ we then have

$$a_k = 0, \quad k = 1, 3, 5... \tag{60}$$

and

$$a_k = -\frac{a_{k-2}}{(k+1)k}, \quad k = 2, 4, 6, ... \tag{61}$$

Then $a_2 = -a_0/3!$, $a_4 = a_0/5!$, ... so letting $k = 2m$ in general

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad k = 1, 2, 3... \tag{62}$$

Choosing $a_0 = 1$, the first solution is

$$y_1(x) = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m}. \tag{63}$$

Second solution: For the second solution in general we require a solution of the form

$$y = c\,y_1(x)\log x + x^{-1/2}\sum_{k=0}^{\infty} b_k x^k. \tag{64}$$

If the roots are of the form $r_2 - r_1 = N$ with $N \in \mathbb{Z}^+$ then the log term is generally required to enable the calculation of $b_N(r_2)$. However, in the present case, we see from (58) that the coefficient of x^{r+1} will vanish for $r_2 = -1/2$ regardless of the value of b_1 (where b_1 replaces a_1 since we are now using $r = r_2$). Hence, the log term is unnecessary in y_2 and $c = 0$. Let's suppose we did not notice this ahead of time and attempted to substitute the general form (64) into (54). Then notice

$$y' = \frac{cy_1}{x} + cy_1' \log x + \frac{d}{dx}\left[x^{-1/2}\sum_{k=0}^{\infty} b_k x^k\right] \tag{65}$$

and

$$y'' = -\frac{cy_1}{x^2} + 2\frac{cy_1'}{x} + cy_1'' \log x + \frac{d^2}{dx^2}\left[x^{-1/2}\sum_{k=0}^{\infty} b_k x^k\right] \tag{66}$$

We then obtain

$$cL[y_1]\log x + 2cxy_1' + L\left[x^{-1/2}\sum_{k=0}^{\infty} b_k x^k\right] = 0. \tag{67}$$

Now we know $L[y_1] = 0$ so this gives

$$L\left[x^{-1/2}\sum_{k=0}^{\infty} b_k x^k\right] = -2cxy_1'. \tag{68}$$

Expanding the left hand side (for convenience we may use (58) and (59) with b_k replacing a_k since we are using $r = r_2$) gives

$$0 \cdot b_0 x^{-1/2} + 0 \cdot b_1 x^{1/2} + (2b_2 + b_0)x^{3/2} + (6b_3 + b_1)x^{5/2} + ... \tag{69}$$

and expanding the right hand side gives

$$-cx^{1/2} + \frac{5}{6}cx^{5/2} - \frac{9}{120}cx^{9/2} \pm ... \tag{70}$$

Equating coefficients give the system of equations

$$\begin{aligned} 0 \cdot b_1 &= -c \\ 2b_2 + b_0 &= 0 \\ 6b_3 + b_1 &= \frac{5}{6}c \\ 12b_4 + b_2 &= 0 \\ &\vdots \end{aligned}$$

So $b_0(r_2)$ and $b_1(r_2)$ are both arbitrary as expected and $c = 0$. The other coefficients are then given by

$$b_{2k} = \frac{(-1)^k b_0}{(2k)!}, \quad b_{2k+1} = \frac{(-1)^k b_1}{(2k+1)!}, \quad k = 1, 2, ... \tag{71}$$

Hence the second solution has the form

$$y_2(x) = x^{-1/2}\left[b_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} + b_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right]. \tag{72}$$

Note that as expected, b_1 just introduces a multiple of $y_1(x)$ so we may choose $b_1 = 0$. Setting the arbitrary constant $b_0 = 1$, the second solution finally becomes

$$y_2(x) = x^{-1/2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}. \tag{73}$$